

# Gauge Threshold Corrections for Local Orientifolds

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**ABSTRACT:** We study gauge threshold corrections for systems of fractional branes at local orientifold singularities and compare with the general Kaplunovsky-Louis expression for locally supersymmetric  $\mathcal{N} = 1$  gauge theories. We focus on branes at orientifolds of the  $\mathbb{C}^3/\mathbb{Z}_4$ ,  $\mathbb{C}^3/\mathbb{Z}_6$  and  $\mathbb{C}^3/\mathbb{Z}'_6$  singularities. We provide a CFT construction of these theories and compute the threshold corrections. Gauge coupling running undergoes two phases: one phase running from the bulk winding scale to the string scale, and a second phase running from the string scale to the infrared. The first phase is associated to the contribution of  $\mathcal{N} = 2$  sectors to the IR  $\beta$  functions and the second phase to the contribution of both  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sectors. In contrast, naive application of the Kaplunovsky-Louis formula gives single running from the bulk winding mode scale. The discrepancy is resolved through 1-loop non-universality of the holomorphic gauge couplings at the singularity, induced by a 1-loop redefinition of the twisted blow-up moduli which couple differently to different gauge nodes. We also study the physics of anomalous and non-anomalous  $U(1)$ s and give a CFT description of how masses for non-anomalous  $U(1)$ s depend on the global properties of cycles.

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## 1. Introduction and Summary of Results

String theory is attractive as a candidate fundamental theory of physics because it has outstandingly soft ultraviolet behaviour. The tower of excited string states tames the divergences that are present in ordinary scattering amplitudes in both quantum field theory and general relativity, returning finite and well-defined answers. Supersymmetry also plays a central role in this process, as although supersymmetry may be broken at long distances, at sufficiently short distances strings see maximal supersymmetry.

Heuristically, field theory divergences are expected to ‘turn off’ somewhere around the string scale as the string-like nature of particles becomes apparent. However it is of great interest to study precisely how divergences are cancelled and the structure of the finite terms that are left over. These terms come from massive string/KK modes and provide a remnant contribution of high-scale physics to low-scale observables. Understanding such threshold corrections is important both from a formal point of view and also when attempting to relate the parameters present in string constructions to the observables of low energy physics.

One of the most important arenas for the study of threshold corrections is the evolution of running gauge couplings. The apparent unification of the gauge couplings at  $M_{GUT} \sim 10^{16}\text{GeV}$  is suggestive of an underlying GUT symmetry broken near the scale  $M_{GUT}$ . Assuming this is not an accident, it is important to understand the significance of  $M_{GUT}$  and how it relates to the compactification parameters. In the perturbative heterotic string, the natural unification scale is the string scale, a factor of around 30 larger than the GUT scale. The original study of threshold corrections was motivated by the possibility that the inclusion of heavy string or Kaluza-Klein modes could remove the discrepancy between the string and unification scales.

More recent model building has occurred in the context of type II string theories (see [1] for a review). One particularly interesting class of models are local or bottom-up constructions. The gauge group and interactions of the Standard Model fields are determined almost entirely by purely local geometry and does not depend on the global properties of the Calabi-Yau. By decoupling the complicated topology of the bulk, local models reduce the geometrical complexity involved in model building. The canonical example of local models is the case of branes at singularities [2], where only the singular geometry is relevant for determining the gauge groups and Yukawa couplings.

The enhanced understanding of moduli stabilisation over the last few years also focuses attention on local models. Moduli stabilisation is best understood in the setting of type IIB flux compactifications. The combination of both full moduli stabilisation and dynamical low scale supersymmetry breaking can be obtained in the LARGE volume scenario [3, 4], which stabilises the bulk at an exponentially large size while keeping blow-up cycles small. In this scenario the observed size of the various Standard Model gauge couplings implies the Standard Model must be realised on a small blow-up cycle, and thus must be represented by a local model.

The above combination of reasons motivates the detailed study of gauge threshold corrections for local models. While the full form of threshold corrections requires a CFT computation, the structure is significantly constrained by effective field theory and in particular by the Kaplunovsky-Louis formula [5, 6]:

$$g_a^{-2}(\Phi, \bar{\Phi}, \mu) = \text{Re}(f_a(\Phi)) + \frac{(\sum_r n_r T_a(r) - 3T_a(G))}{8\pi^2} \ln\left(\frac{M_P}{\mu}\right) + \frac{T(G)}{8\pi^2} \ln g_a^{-2}(\Phi, \bar{\Phi}, \mu) \\ + \frac{(\sum_r n_r T_a(r) - T(G))}{16\pi^2} \hat{K}(\Phi, \bar{\Phi}) - \sum_r \frac{T_a(r)}{8\pi^2} \ln \det Z^r(\Phi, \bar{\Phi}, \mu). \quad (1.1)$$

Here  $g_a^{-2}(\Phi, \bar{\Phi}, \mu)$  is the physical coupling,  $f_a(\Phi)$  the holomorphic coupling,  $\mu$  the energy scale, and  $\Phi$  light uncharged moduli superfields.  $\hat{K}$  is the moduli Kähler potential and  $Z^r$  are the matter field kinetic terms. Equation (1.1) simplifies considerably for local models. The requirement that physical Yukawa couplings  $\hat{Y}_{\alpha\beta\gamma} = \frac{e^{\mathcal{K}/2} Y_{\alpha\beta\gamma}}{\sqrt{Z_\alpha Z_\beta Z_\gamma}}$  do not depend on the bulk volume strongly constrains the dependence of the matter metrics  $Z_\alpha$  on  $\mathcal{V}$ . As  $\mathcal{K} = -2 \ln \mathcal{V}$  and  $Y_{\alpha\beta\gamma}$  is independent of  $\mathcal{V}$ , this implies  $Z_\alpha = 1/\mathcal{V}^{2/3}$ .

For local models equation (1.1) therefore becomes

$$g_a^{-2}(\mu) - \frac{T(G)}{8\pi^2} \ln g_a^{-2}(\mu) = \text{Re}(f_a(\Phi)) + \beta_a \ln\left(\frac{(RM_s)^2}{\mu^2}\right), \quad (1.2)$$

where  $R$  is the bulk radius  $R = \mathcal{V}^{1/6}$ . For universal  $f_a(\Phi)$  this implies that the unification scale is given by  $M_X = RM_s \gg M_s$  for  $R \gg 1$ . This is quite surprising as the scale  $RM_s$  depends on the bulk whereas naively local models are insensitive to the bulk. However the interpretation of the field theory formula (1.1) can be subtle due to field redefinitions and chiral/linear multiplet dualities. Eq. (1.2) therefore motivates a detailed CFT study of the threshold corrections in order to understand the physics of this apparent unification at  $M_X \gg M_s$ .

This study was initiated in [7] where threshold corrections were studied for branes at orbifold singularities. In [7] systems of D3 branes at orbifold singularities were found to exhibit unification at  $RM_s$ , whereas a D3/D7 system gave unification at  $M_s$  in apparent disagreement with (1.2). In this paper we continue this analysis, focusing our attention on orientifolded singularities. We shall resolve the discrepancy encountered in [7] and obtain a precise understanding of when running starts at  $M_s$ , when running starts at  $M_X$ , and when a combination of the two applies. Full agreement with (1.1) is found after incorporating the effects of one-loop redefinitions of the moduli superfields. In [8] we will further apply this understanding of threshold corrections to local IIB/F-theory GUTs [9, 10] which exist in the geometric regime where the CFT computations cannot be performed.

As the actual calculations are rather technical, in the remainder of this introduction we shall summarise the methodology and results of this paper.

## Summary of Results

For models at orbifold/orientifold singularities, the gauge groups comes from fractional branes, whose geometric interpretation is as magnetised branes or antibranes wrapping

collapsed cycles. The number and type of the possible fractional branes is determined by the orbifold. Each fractional brane corresponds to a node of the quiver and the gauge coupling on each brane is

$$f_a = S + s_k^a M_k, \quad (1.3)$$

where  $S = \frac{1}{g_s} + ic_0$  is the axio-dilaton and  $M_k$  corresponds to the twisted blow-up moduli.<sup>1</sup> The  $s_{ak}$  encode the charges of each fractional brane under the RR fields induced by the Chern-Simons term in the action. The orientifold also introduces fractional O-planes which are likewise wrapped on the collapsing cycles and contribute to the RR charges along the collapsed cycles. An orientifolded singularity imposes relationships between the different fractional branes and projects out some of the twisted moduli from the orbifold.

Consistency of the theory requires cancellation of all RR tadpoles. The tadpoles in local models come in several kinds related to the geometry of the singularity. First, there are purely local tadpoles. These correspond to 2/4-cycles where both the cycle and its dual cycle are defined in the local geometry. These local cycles are the unique supersymmetric cycles within this homology class. In orbifold parlance, these are fully twisted  $\mathcal{N} = 1$  sectors. Heuristically speaking, a tadpole along such a cycle has nowhere to go: it cannot escape to infinity and must be cancelled locally. Cancellation of  $\mathcal{N} = 1$  tadpoles corresponds to cancellation of gauge anomalies in the effective field theory.

There are also global tadpoles. These correspond to RR charges which are sourced locally but can be cancelled globally. Geometrically, these correspond to cycles where a 2- or 4-cycle can be defined locally, but globally may either be trivial or there may exist other calibrated cycles in the same homology class. Examples of these are given by the del Pezzo singularities:  $dP_n$  has 1 4-cycle and  $(n + 1)$  2-cycles, of which up to  $n$  of the 2-cycles may be globally trivial. In orbifold parlance, these represent partially twisted  $\mathcal{N} = 2$  sectors, and so (for example) the  $\Delta_{27}$  orbifold (which is a limit of the  $dP_8$  singularity) has 8  $\mathcal{N} = 2$  sectors. Such tadpoles need not be cancelled locally and do not constrain the allowed numbers of branes. Finally, there is also the untwisted  $\mathcal{N} = 4$  sector, associated to the dilaton tadpole and corresponding to the total number of branes at the singularity.

There exist various fractional brane configurations cancelling  $\mathcal{N} = 1$  tadpoles. The choice of configuration determines the gauge groups and massless spectrum and thus the IR beta functions. The spectrum also contains heavy string and KK modes, loops of which give rise to threshold corrections. The threshold corrections  $\Delta_a(M, \bar{M})$  are moduli-dependent and can be defined by

$$\frac{1}{g_a^2(\mu)} = \frac{1}{g_a^2} \Big|_0 + \beta_a \ln \left( \frac{M_s^2}{\mu^2} \right) + \Delta_a(M, \bar{M}). \quad (1.4)$$

In this notation, threshold corrections represent the difference between the actual low-energy couplings and those obtained by field theory running starting from the string scale.

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<sup>1</sup>For some singularities the different nodes can have non-universal couplings to the dilaton. In such cases the use of ‘unification’ in this paper would refer to the gauge couplings having the ratios given by their dilaton coupling.

Threshold corrections are computed via an open string one-loop diagram, which via open-closed duality is equivalent to a closed string tree level diagram. This relationship implies that ultraviolet finiteness of the threshold corrections is equivalent to infrared finiteness in closed string channel, namely tadpole cancellation.

In this paper we use the background field approach to compute the threshold corrections [11–14, 17]. This involves turning on a background spacetime magnetic field  $B$  in a generator of the gauge group  $U(N_a)$  for which we want to compute the threshold corrections. The one-loop vacuum energy in the presence of this background field can be expanded as

$$\Lambda = \Lambda_0 + \left(\frac{B}{4\pi^2}\right)^2 \Lambda_2^a + \left(\frac{B}{4\pi^2}\right)^4 \Lambda_4^a + \dots \quad (1.5)$$

The gauge threshold corrections can be extracted by analysis of the  $\mathcal{O}(B^2)$  term. In a consistent theory the  $B^2$  term is finite for non-abelian background fields, and UV finiteness of the  $\mathcal{O}(B^2)$  amplitudes provides another way to compute the tadpole and anomaly conditions. Terms of  $\mathcal{O}(B^4)$  are generally ultraviolet divergent. These divergences correspond in closed string channel to on-shell exchange of massless string states and the coefficients of such terms can be used to extract the tree-level couplings of gauge groups to the local twisted closed string moduli.

The term  $\Lambda_2$  can be written

$$\Lambda_2 = (8\pi^2) \int_0^{1/\mu^2} \frac{dt}{t} \Delta_a(t), \quad (1.6)$$

where  $\mu$  is the IR regulator.  $\Delta_a$  is a partition function of the schematic form  $\text{STr}(e^{-m^2 t})$ . In the infrared limit  $t \rightarrow \infty$ ,  $\Delta_a(t) \rightarrow \beta_a$  reproducing the field theory beta functions. In the UV limit  $t \rightarrow 0$ , for non-abelian groups  $\Delta_a(t) \rightarrow 0$  reflecting the finiteness of the theory. Note for abelian groups ultraviolet divergences may occur in  $\Lambda_2$  via the  $B_{\mu\nu} \wedge F_2$  Green-Schwarz coupling.

The threshold corrections are encapsulated in the precise way  $\Delta_a(t)$  vanishes in the regime  $t \lesssim 1$ . The actual computation of one-loop threshold computations therefore reduces to computing the string partition function on the local orbifold/orientifold geometry. For  $\mathbb{Z}_n$  orbifold/orientifold singularities, the partition function involves a projection

$$\Delta_a(t) = \text{STr} \left( \frac{(1 + \theta + \theta^2 + \dots + \theta^{N-1})}{N} e^{-m^2 t} \right) \equiv \frac{1}{N} \sum_{k=0}^{N-1} \Delta_a^{(k)}(t).$$

The sector  $\Delta_a^{(k)}$  is called an  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  sector depending in whether  $\theta^k$  fixes all 2-tori ( $\mathcal{N} = 1$  sector) or leaves one torus unfixed ( $\mathcal{N} = 2$  sector).  $\Delta_a^{(0)}(t)$  represents the only  $\mathcal{N} = 4$  sector and vanishes consistent with the non-renormalisation properties of  $\mathcal{N} = 4$  supersymmetry. In general  $\Delta_a^{(k)}$  is non-zero for both  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sectors and in the  $t \rightarrow \infty$  limit we have  $\Delta_a^{(k)} \rightarrow \beta_a^{(k)}$  with  $\beta_a = \frac{1}{N} \sum_{k=1}^{N-1} \beta_a^{(k)}$ .

The threshold corrections are encoded in the  $t \rightarrow 0$  behaviour of  $\Delta_a^{(k)}$ . Let us state the schematic form of these and then explain the results.

$$\Delta_a^{(k)} = \begin{cases} \beta_a^{(k)} \Theta \left[ t - \frac{1}{M_s^2} \right] + \text{small}, & \mathcal{N} = 1 \text{ sector} \\ \beta_a^{(k)} \Theta \left[ t - \frac{1}{(RM_s)^2} \right] + \text{small}, & \mathcal{N} = 2 \text{ sector} \\ 0 & \mathcal{N} = 4 \text{ sector} \end{cases}, \quad (1.7)$$

where  $\Theta$  is the Heaviside theta function and  $R$  the bulk radius. The gauge coupling running therefore takes the form

$$\frac{1}{g^2}(\mu) = \frac{1}{g^2} \Big|_0 + \beta_a \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_a^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right). \quad (1.8)$$

The form of (1.7) can be understood by reference to the above picture of cycles and their geometries. In each sector, the gauge coupling runs up to a certain energy scale and is then cut off. The energy scale of the cutoff is determined by the mass of the string states necessary to obtain tadpole cancellation. For purely local cycles ( $\mathcal{N} = 1$  sectors), tadpole cancellation is local and occurs once string scale states are included. This leads to an effective cutoff on  $\Delta_{\mathcal{N}=1}^{(k)}$  at  $t \sim \frac{1}{M_s^2}$ . For  $\mathcal{N} = 2$  sectors, tadpole cancellation does not occur locally and instead requires knowledge of the bulk geometry. From an open string perspective this requires the inclusion of brane-brane winding modes that reach out into the bulk.  $\mathcal{N} = 2$  supersymmetry prevents the (non-BPS) string-scale oscillator tower from contributing to gauge coupling running and field theory running is maintained until winding modes comes in at a scale  $RM_s$ , when  $\Delta_{\mathcal{N}=2}^{(k)}$  is finally cut off. We also note that for branes at singularities there are no charged KK modes that can contribute to the threshold corrections.

The net effect is that  $\mathcal{N} = 1$  sectors give field theory running up to the string scale, where they are cut off, while  $\mathcal{N} = 2$  sectors give field theory running up to the winding string scale.  $\mathcal{N} = 4$  sectors give no contribution due to the effective maximal supersymmetry that is present. Physics close to the cutoffs introduces small additional corrections, that is however suppressed compared to the large  $R$  enhanced terms. Some discussions of these additional corrections can be found in [15, 16].

For the case of D3 branes at orbifold singularities, tadpole cancellation required the coefficient of all  $\mathcal{N} = 1$  sectors to vanish, and the  $\beta$ -functions arose entirely from the  $\mathcal{N} = 2$  sectors (even though the low-energy spectrum is chiral and  $\mathcal{N} = 1$  supersymmetric). In this case all running is from the winding mode scale, straightforwardly consistent with the Kaplunovsky-Louis formula. For the case of both orientifolded singularities and D3/D7 systems, both  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sectors contribute to the  $\beta$ -functions. In general the  $\mathcal{N} = 1$  contributions to the  $\beta$  functions are not universal and there is no apparent unification scale. To reconcile this with the Kaplunovsky-Louis formula, recall the form of the tree level holomorphic gauge coupling (1.3) which shows that the gauge coupling can receive a non-universal correction from a vev for the  $M_k$  superfields  $\langle \text{Re} M_k \rangle \neq 0$ . The string calculation is performed in the orbifold limit which we denote by the string real twisted

mode  $\langle m_k \rangle = 0$ . At tree level the two fields coincide with  $\text{Re}M_k = m_k$ . However at 1-loop the relation is modified

$$\text{Re}(M_k) = m_k - \alpha_k \ln R^2, \quad (1.9)$$

where  $\alpha_k$  is some constant, such that at the orbifold point

$$s_k^a \text{Re}M_k = -\beta_a^{(k)} \ln \left( \frac{M_X^2}{M_s^2} \right). \quad (1.10)$$

This exactly accounts for the discrepancy between the string calculation (1.8) and the KL formula (1.2).

This field redefinition is familiar from heterotic and type I orbifolds [17, 18]. It arises because  $M_k$  are components of a chiral multiplet while  $m_k$  is the scalar component of a linear multiplet. The dualisation procedure receives a 1-loop correction (1.9) with the correction proportional to the correction induced at 1-loop to the  $\beta$  functions. Consistency with (1.2) requires that the couplings  $s_k^a$  in (1.10) are proportional to  $\beta_a^{(k)}$ , a fact we explicitly compute in section 4.

The redefinition (1.9) is related to the  $\beta$ -function contribution associated to the  $\mathcal{N} = 1$  twisted mode. Such contributions are present for both orientifold and D3/D7 singularities. For branes at orientifolded singularities, there are contributions to  $\mathcal{N} = 1$  sectors from combining the Möbius and Annulus diagrams. For D3/D7 systems, the D3/D3 and D3/D7 diagrams combine to give the  $\mathcal{N} = 1$  contributions. For branes at orbifold singularities, there are only D3/D3 diagrams and so  $\Delta_a^{\mathcal{N}=1}$  has to vanish in order to enforce UV tadpole cancellation.

Once the one-loop redefinition (1.9) is carried out the resulting gauge couplings agree with the Kaplunovsky-Louis formula. In this case the holomorphic gauge couplings  $f_a(\Phi)$ , which were universal at tree level due to the vanishing of  $M_k$ , become non-universal at one loop. The apparent unification of physical couplings at  $RM_s$ , which is present for models of D3s at orbifold singularities, is not present for D3/D7 models or for D3s at orientifolded singularities.

In summary, physical gauge couplings run from  $RM_s$  if the  $\beta$  functions are sourced only from  $\mathcal{N} = 2$  sectors and from  $M_s$  if  $\beta$  functions are sourced only from  $\mathcal{N} = 1$  sectors. In the case that both  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sectors contribute then  $\mathcal{N} = 2$  running starts at  $RM_s$  with a significant, generically non gauge universal, shift in the effective  $\beta$ -functions at  $M_s$  as the  $\mathcal{N} = 1$  sectors add their contribution to running from the scale  $M_s$ .<sup>2</sup>

The organisation of this paper is as follows. The paper studies branes at orientifolds of the  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$  and  $\mathbb{Z}_6'$  singularities. As far as we aware the field theory on such singularities has not been explicitly constructed before and so in section 2 we first provide a CFT derivation of the gauge groups and spectrum. We focus particularly on the  $\mathbb{Z}_4$  case that will serve as our main example throughout this paper. In section 3 we describe the computation of

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<sup>2</sup>It is not clear whether operational meaning can be applied to a gauge coupling at an energy scale above  $M_s$ . An unambiguous statement is to instead say that the low-energy gauge couplings, which are well-defined, behave as if they have been run down from a scale  $RM_s$ .



threshold corrections and in section 4 we describe the matching to the effective field theory structure. In section 5 we summarise results for the  $\mathbb{Z}_6$  and  $\mathbb{Z}'_6$  orientifolds. In appendix A we study anomalous and non-anomalous  $U(1)$ s in local models and in particular the Green-Schwarz mechanism within the local model and its global completion. In appendix B we derive general expressions for the tadpole amplitudes. In appendix C we calculate general expressions for the magnetised amplitudes. In appendix D we discuss in more detail the dualisation procedure between chiral and linear multiplets and the 1-loop corrections this receives. In appendix E we give some useful expressions and transformation properties for the  $\vartheta$  functions.

## 2. Orientifold constructions

We start by describing the orientifold constructions that will be used for our calculations. The CFT construction of orientifolds is standard and more details can be found in [2, 20–22] for example.

### 2.1 Orientifolds of orbifold singularities

We start with a local orbifold singularity  $\mathbb{C}^3/\mathbb{Z}_N$  where the orbifold action is generated by the element  $\theta$  acting as  $\theta : z_i \rightarrow \exp(2\pi i \theta_i) z_i$ ,  $i = 1, 2, 3$ , with components  $\theta_i$  running over the local complex co-ordinates of the internal manifold  $z_i$ . The orbifold group is formed of elements  $\theta^k$  produced by  $k$  applications of  $\theta$ . The generating orbifold element also has an action on the Chan-Paton (CP) indices of the open strings,

$$\gamma_\theta = \text{diag}(\mathbf{1}_{n_0}, \alpha \mathbf{1}_{n_1}, \dots, \alpha^N \mathbf{1}_{n_N}) , \quad (2.1)$$

where  $\alpha$  denotes the  $N^{\text{th}}$  root of unity and  $\mathbf{1}_{n_i}$  denotes the  $n_i \times n_i$  unit matrix. The integers  $n_i$  correspond to the number of fractional branes on each node of the quiver.

The resulting gauge theory is an  $\mathcal{N} = 1$  supersymmetric  $\prod_i^N U(n_i)$  gauge theory and the massless fermionic open string spectrum is given by the CP elements that satisfy the orbifold projection

$$\lambda = e^{2\pi i (\sum_i \theta_i s_i)} \gamma_\theta \lambda \gamma_\theta^{-1} \equiv e^{2\pi i \underline{\theta} \cdot \mathbf{s}} \gamma_\theta \lambda \gamma_\theta^{-1} . \quad (2.2)$$

Here  $\lambda$  denotes the  $M \times M$  (with  $M = \sum_i n_i$ ) CP matrix. The vector  $\mathbf{s}$  denotes the spin of the RR ground states and its elements take the values  $s_i = \pm 1/2$ . The GSO projection requires the number of negative spins to be even.

The closed string twisted spectrum gives a single complex scalar field per element in  $\mathbb{Z}_N$ . The twisted sectors are labelled according to the amount of supersymmetry preserved, namely  $\mathcal{N} = 4$ ,  $\mathcal{N} = 2$  and  $\mathcal{N} = 1$  for the cases that three complex directions, one complex direction and no complex directions are left fixed by the geometric orbifold twist. An important fact is that  $\mathcal{N} = 1$  closed string modes are restricted to lie on the singularity, while  $\mathcal{N} = 2$  modes can propagate into the bulk along the complex direction that is left fixed.

We can orientifold the singularity by introducing an orientifold involution

$$\Omega' = \Omega IR(-1)^{F_L} . \quad (2.3)$$

Here  $\Omega$  is world-sheet parity inversion.  $I$  is spatial inversion given by the rotation  $\underline{\theta} = (1/2, 1/2, 1/2)$ .  $R$  is a further spatial action whose geometric action must square to an element of the orbifold group

$$R^2 = \theta^l , \quad (2.4)$$

for some  $l$ . This ensures that the orientifold is indeed a good involution of the orbifolded space. The action of the orientifold on the CP indices is

$$\Omega' : \lambda \rightarrow \gamma_{\Omega'} \lambda^T \gamma_{\Omega'}^{-1} . \quad (2.5)$$

Since  $\Omega'$  must square to an element of the orbifold group we require

$$\gamma_{\Omega'} \gamma_{\Omega'}^{-T} = \pm (\gamma_{\theta})^l , \quad (2.6)$$

for the same  $l$  as in (2.4). Note the  $+$  sign in (2.6) corresponds to what is usually termed the  $SO$  projection (rather than the  $Sp$ ), and we keep this sign choice for the rest of the paper. We also generally denote  $(\gamma_{\theta})^k \gamma_{\Omega'} = \gamma_{\Omega'_k}$  giving

$$\gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} = + (\gamma_{\theta})^{2k+l} . \quad (2.7)$$

Together, the orbifold group and the orientifold action form the orientifold group

$$\{1, \theta, \theta^2, \dots, \theta^{N-1}, \Omega', \Omega' \theta, \dots, \Omega' \theta^{N-1}\} . \quad (2.8)$$

The orientifold planes present in the construction are determined by the fixed point set of the spatial involution  $IR$  quotiented by the action of the orbifold group. In IIB there are two basic types of orientifold projection,  $O3/O7$  and  $O5/O9$ . These have

$$O3/O7 : \quad IR : J \rightarrow J, \quad IR : \Omega \rightarrow -\Omega,$$

$$O5/O9 : \quad IR : J \rightarrow J, \quad IR : \Omega \rightarrow \Omega.$$

As we are interested in local models we will require  $IR$  to satisfy the  $O3/O7$  conditions. We will further require that on the non-compact orbifold  $\mathbb{C}^3/\mathbb{Z}_n$  the only fixed point of  $IR$  is the origin. This will ensure that only  $O3$  planes are present.

Given the orientifold action, the resulting massless spectrum is a projection from the orbifold spectrum which for the fermionic open string modes reads

$$\lambda = -e^{2\pi i(\sum_i R_i s_i)} \gamma_{\Omega'} \lambda^T \gamma_{\Omega'}^{-1} . \quad (2.9)$$

## 2.2 Tadpole amplitudes

The act of orientifolding introduces O-planes which source RR tadpoles. Consistency requires the introduction of branes to cancel these tadpoles. For orientifolded singularities the O-planes wrap the collapsed cycle and carry RR charge under the various cycles of the singularity. Tadpoles for  $\mathcal{N} = 1$  fields must be cancelled locally and correspond to field theory gauge anomalies.  $\mathcal{N} = 2$  tadpoles need not be cancelled locally since a net source of a  $\mathcal{N} = 2$  closed string mode can be balanced by sinks in the bulk space. The tree-level  $\mathcal{N} = 1$  closed string tadpoles can be calculated by studying the quadratic divergences of one-loop open string amplitudes given by the annulus, Mobius strip, and Klein bottle (labelled  $\mathcal{A}$ ,  $\mathcal{M}$  and  $\mathcal{K}$  respectively). The methods to compute these tadpoles and generate consistent brane configurations are well known. Here we simply state results and leave the details to Appendix B. The amplitudes all diverge linearly with the closed string cylinder length parameter  $l$  and in the open string UV limit  $l \rightarrow \infty$  read<sup>3</sup>

$$\mathcal{A}_{\mathcal{N}=1}^{(k)} \xrightarrow[l' \rightarrow \infty]{UV} - \int_{l'}^{\infty} \frac{dl}{4\pi^2} \frac{1}{4} \text{Tr} [\gamma_k] \text{Tr} [\gamma_k^{-1}] \prod_{i=1}^3 \left| 2 \sin(\pi \theta_i^k) \right|, \quad (2.10)$$

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow[l' \rightarrow \infty]{UV} \int_{l'}^{\infty} \frac{dl}{4\pi^2} \left[ 2 \text{Tr} [\gamma_{\Omega'_k} \gamma_{\Omega'^{-T}}] \prod_{i=1}^3 s_i \left( 2 \sin(\pi R_i^k) \right) \right], \quad (2.11)$$

$$\mathcal{K}_{0,\mathcal{N}=1}^{(k)} + \mathcal{K}_{2,\mathcal{N}=1}^{(k)} \xrightarrow[l' \rightarrow \infty]{UV} - \int_{l'}^{\infty} \frac{dl}{4\pi^2} 4 \left[ \prod_{i=1}^3 \left| \frac{\sin(\pi R_i^k)}{\cos(\pi R_i^k)} \right| + (-1)^M \prod_{i=1}^3 (-1)^{\delta_i} \left| \frac{\sin(\pi R_i^k)}{\cos(\pi R_i^k)} \right|^{\delta_i} \right].$$

where  $s_i = \text{sgn}[\sin(2\pi R_i^k)]$  and  $R_i^k = \theta_i^k + R_i$ . The amplitude  $\mathcal{K}_2^{(k)}$  corresponds to the partition function for the closed string  $\mathbb{Z}_2$  twisted sector and thus is only present for even orientifolds. For this case

$$\delta_i = \begin{cases} 0 & \text{if } \theta_i^{N/2} \bmod 1 = 1/2 \\ 1 & \text{otherwise} \end{cases}. \quad (2.12)$$

The contributions from other closed string twisted sectors vanish as they are exchanged by the orientifold action.

The superscript  $k$  on the amplitudes and angles denotes the element in the orientifold group, with rotation angles  $R^k$  coming from elements involving  $\Omega' \theta^k$ .<sup>4</sup> The tadpole constraint is that the sum over all fully twisted elements  $\theta^k$  and  $R^k$  corresponding to any single closed string twisted mode should vanish. Partially twisted ( $\mathcal{N} = 2$ ) tadpoles are

<sup>3</sup>Throughout the paper we often switch between the open string loop channel and the closed string tree channel. By the UV limit we refer to the open string UV limit which is the closed string IR limit.

<sup>4</sup>The  $(-1)^M$  factor in (2.11) is associated with the sign of the action of the orientifold on the NS-NS ground state in the  $\mathbb{Z}_2$  twisted sector:

$$\Omega IR(-1)^{FL} \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \otimes \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle_{NS-NS, \mathbb{Z}_2} = (-1)^M \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \otimes \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle_{NS-NS, \mathbb{Z}_2},$$

and for the orientifolds in this paper takes the value of  $M = 0$  for the  $\mathbb{Z}_4$  and  $\mathbb{Z}_6'$  cases and  $M = 1$  for the  $\mathbb{Z}_6$  case.

not required to vanish in a local model. However in a fully global model such tadpoles must vanish once summed over all global sectors.

### 2.3 The canonical example: $\mathbb{Z}_4$

We now develop our basic example, the orientifold of the  $\mathbb{Z}_4$  orbifold singularity. This model is used throughout the paper as the canonical example exhibiting the physics we discuss. We study two further constructions based on  $\mathbb{Z}_6$  and  $\mathbb{Z}'_6$  singularities in section 5.

The  $\mathbb{Z}_4$  orbifold action is generated by  $\theta = (1/4, 1/4, -1/2)$ . We take the orientifold spatial action to be  $R = (1/8, 1/8, -1/4)$ . The orientifold group is therefore

$$\left\{ (0, 0, 0), \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, -1\right), \left(\frac{3}{4}, \frac{3}{4}, -\frac{3}{2}\right), \right. \\ \left. \Omega I \left(\frac{1}{8}, \frac{1}{8}, -\frac{1}{4}\right), \Omega I \left(\frac{3}{8}, \frac{3}{8}, -\frac{3}{4}\right), \Omega I \left(\frac{5}{8}, \frac{5}{8}, -\frac{5}{4}\right), \Omega I \left(\frac{7}{8}, \frac{7}{8}, -\frac{7}{4}\right) \right\}. \quad (2.13)$$

It is easy to verify that  $IR\Omega = -\Omega$  as required for an  $O3/O7$  projection. As all elements involving  $\Omega$  have fully twisted spatial parts there are no  $O7$  planes and all  $O3$  planes are located at the origin. It is this property that makes the model purely local as there are no branes or orientifold planes extending from the singularity into the bulk.

We take the orbifold generating element

$$\gamma_\theta = \text{diag}(\mathbf{1}_{n_0}, \alpha \mathbf{1}_{n_1}, \alpha^2 \mathbf{1}_{n_2}, \alpha^3 \mathbf{1}_{n_3}) \quad \text{with } \alpha = e^{\pi i/2}, \quad (2.14)$$

and impose  $n_1 = n_3$ . For the orientifold action we take

$$\gamma_{\Omega'} = \begin{pmatrix} \mathbf{1}_{n_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^{3/2} \mathbf{1}_{n_3} \\ 0 & 0 & \alpha \epsilon_{n_2} & 0 \\ 0 & \alpha^{1/2} \mathbf{1}_{n_1} & 0 & 0 \end{pmatrix}, \quad (2.15)$$

with  $\epsilon$  denoting the anti-symmetric matrix with unit off-diagonal entries. These matrices satisfy the constraint (2.6) so that  $\Omega'$  has a well-defined  $\mathbb{Z}_2$  action on the orbifold Hilbert space.

Calculating the tadpoles using (2.10-2.11) leads to

$$\text{Tr}[\gamma_\theta] + 4 = 0. \quad (2.16)$$

The tadpole constraints impose the condition

$$n_2 = n_0 + 4. \quad (2.17)$$

The massless fermionic spectrum of the theory can be calculated using (2.9) which gives the matter content shown in table 1. The gauge group is

$$G = SO(n_0) \times U(n_1) \times Sp(n_2) \equiv SO(n_0) \times U(n_1) \times Sp(n_0 + 4). \quad (2.18)$$

Multiplicity	Representation		
	$SO(n_0)$	$SU(n_1)$	$Sp(n_2)$
2	$n_0$	$\bar{n}_1$	1
2	1	$n_1$	$n_2$
1	$n_0$	1	$n_2$
1	1	$A_{n_1}$	1
1	1	$\bar{S}_{n_1}$	1

**Table 1:** Field content and representations for  $\mathbb{Z}_4$  orientifold. The  $n_i$  denote the fundamental representation and  $S$  and  $A$  denote symmetric and anti-symmetric representations respectively.

The non-abelian anomalies of the theory are equivalent to the tadpole constraint (2.17). We are also interested in the field theory  $\beta$ -functions for the gauge groups. After imposing anomaly cancellation (2.17) these read

$$\beta_{SU(n_1)} = \frac{1}{16\pi^2} (-2n_1 + 2n_0 + 4) , \quad (2.19)$$

$$\beta_{SO(n_0)} = \frac{1}{16\pi^2} (-2n_0 + 2n_1 + 10) , \quad (2.20)$$

$$\beta_{Sp(n_2)} = \frac{1}{16\pi^2} (-2n_0 + 2n_1 - 18) . \quad (2.21)$$

We will use this orientifold of the  $\mathbb{C}^3/\mathbb{Z}_4$  singularity as the principal example for our study of the physics of string threshold corrections to field theory running.

### 3. Threshold corrections: the string calculation

To calculate the string threshold corrections we use the background field method [11–13, 17]. The calculation proceeds by turning on a background magnetic field in the non-compact dimensions then calculating the resulting one-loop vacuum energy. We write the background magnetic field as  $F_{23}^a = BQ_a$  where  $a$  denotes the gauge group,  $Q_a$  is the generator inside the gauge group, the indices 23 denote spatial directions, and  $B$  is the magnitude of the field. Recall that the one-loop vacuum energy,  $\Lambda$ , takes the form

$$\Lambda = \Lambda_0 + \left(\frac{B}{4\pi^2}\right)^2 \Lambda_2^a + \left(\frac{B}{4\pi^2}\right)^4 \Lambda_4^a + \dots \quad (3.1)$$

The contribution  $\Lambda_0$  vanishes in a supersymmetric vacuum. The coefficient  $\Lambda_2^a$  gives the full one-loop threshold corrections<sup>5</sup>

$$\frac{1}{g_a^2} \Big|_{1\text{-loop}} = \frac{1}{g_a^2} \Big|_{\text{tree-level}} + \frac{\Lambda_2^a}{8\pi^2} . \quad (3.2)$$

---

<sup>5</sup>Note that  $\Lambda_2^a$  is sensitive to the Lagrangian terms  $F \wedge \star F$  and  $C_2 \wedge F$ , but not  $F \wedge F$ . This is because we have turned on the magnetic field along only two space-time directions. This is why it gives exactly the gauge coupling (up to a possible Green-Schwarz contribution which we discuss in section 4.).

### 3.1 Magnetised amplitudes

In this section we are primarily concerned with calculating  $\Lambda_2^a$  and extracting its IR and UV behaviour. The contributing amplitudes to  $\Lambda_2^a$  are Annulus and Mobius amplitudes (since the torus and Klein bottle do not couple to the gauge field) so that

$$\left(\frac{B}{4\pi^2}\right)^2 \Lambda_2^a = (\mathcal{A}^a + \mathcal{M}^a)|_{B^2} . \quad (3.3)$$

The full calculation is presented in Appendix C, to which we refer for more details regarding the expressions, and in this section we draw on the key results.

The fully twisted ( $\mathcal{N} = 1$ ) D3-D3 Annulus amplitude in the background of a magnetic field is given by [7]

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=1}^{(k)} = & - \int_0^\infty \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left( (\gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1}) \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right) \\ & \times \prod_{i=1}^3 \frac{(-2 \sin(\pi \theta_i^k)) \vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + \theta_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + \theta_i^k \end{smallmatrix} \right]}, \end{aligned} \quad (3.4)$$

where we decompose the amplitude into its orbifold sectors

$$\mathcal{A} = \frac{1}{N} \sum_{k=0}^N \mathcal{A}^{(k)} , \quad (3.5)$$

and the subscript  $\mathcal{N} = 1$  denotes that this result applies to orbifold sectors that are fully twisted. Here we denote the charges of the left and right ends of the string as  $q_1$  and  $q_2$  respectively, and write  $\beta_1 = Bq_1$  and  $\beta_2 = Bq_2$ . Neutral strings have opposite charges on their ends. We also define

$$\epsilon = \frac{1}{\pi} (\arctan \beta_1 + \arctan \beta_2) . \quad (3.6)$$

Similarly we also have the magnetised Mobius  $\mathcal{N} = 1$  amplitude

$$\begin{aligned} \mathcal{M}_{\mathcal{N}=1}^{(k)} = & 2 \int_0^\infty \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \times \\ & \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left[ \frac{i}{2\pi^2} \beta \gamma_{\Omega_k} \gamma_{\Omega_k}^{-T} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right] \prod_{i=1}^3 \left( -2 \sin(\pi R_i^k) \right) \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + R_i^k \end{smallmatrix} \right]} . \end{aligned} \quad (3.7)$$

As left and right ends of the string are identified there is only one charge for the string denoted  $q$  with  $\beta = Bq$  and  $\epsilon = \frac{2}{\pi} \arctan \beta$ . The angles are defined as

$$R_i^k = \theta_i^k + R_i . \quad (3.8)$$

We are interested in the IR and UV behaviour of the  $B^2$  terms in the amplitudes. This calculation was performed in [7] for the Annulus, and the Mobius strip can be calculated in the same way. The results are that the IR limit of the  $\mathcal{N} = 1$   $B^2$  part of the amplitudes is given by

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=1}^{(k)} \xrightarrow[t' \rightarrow \infty]{IR} & - \left( \frac{B}{4\pi^2} \right)^2 \int_{t'}^{\infty} \frac{dt}{2t} \text{Tr} \left[ \frac{1}{2} (q_1^2 \gamma_k \otimes \gamma_k^{-1} + 2q_1 \gamma_k \otimes \gamma_k^{-1} q_2 + \gamma_k \otimes \gamma_k^{-1} q_2^2) \right] \\ & \times \sum_{i=1}^3 \frac{\cos(\pi \theta_i^k)}{\sin(\pi \theta_i^k)} \prod_{i=1}^3 \left( -2 \sin(\pi \theta_i^k) \right) . \\ \mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow[t' \rightarrow \infty]{IR} & - \left( \frac{B}{4\pi^2} \right)^2 \int_{t'}^{\infty} \frac{dt}{2t} \text{Tr} \left[ -2q^2 \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \right] \sum_{i=1}^3 \frac{\cos(\pi R_i^k)}{\sin(\pi R_i^k)} \prod_{i=1}^3 \left( -2 \sin(\pi R_i^k) \right) \end{aligned} \quad (3.9)$$

We can also extract the UV limit by going to the dual closed string channel with cylinder length parameter  $l = 1/t$  for the Annulus and  $l = 1/4t$  for the Mobius, which gives

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=1}^{(k)} \xrightarrow[l' \rightarrow \infty]{UV} & - \left( \frac{B}{4\pi^2} \right)^2 \int_{l'}^{\infty} dl \text{Tr} \left[ \frac{1}{2} (q_1^2 \gamma_k \otimes \gamma_k^{-1} + 2q_1 \gamma_k \otimes \gamma_k^{-1} q_2 + \gamma_k \otimes \gamma_k^{-1} q_2^2) \right] \prod_{i=1}^3 \left| 2 \sin \pi \theta_i^k \right| , \\ \mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow[l' \rightarrow \infty]{UV} & - \left( \frac{B}{4\pi^2} \right)^2 \int_{l'}^{\infty} dl \text{Tr} \left[ 4q^2 \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \right] \prod_{i=1}^3 s_i \left( -2 \sin(\pi R_i^k) \right) . \end{aligned} \quad (3.10)$$

The  $\mathcal{N} = 4$  untwisted sectors do not contribute to the  $B^2$  terms due to supersymmetry. There are also  $\mathcal{N} = 2$  contributions and these take the exact form

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=2}^{(k)} & = \left( \frac{B}{4\pi^2} \right)^2 \int_0^{\infty} \frac{dt}{2t} \text{Tr} \left[ (q_1^2 \gamma_k \otimes \gamma_k^{-1} + 2q_1 \gamma_k \otimes \gamma_k^{-1} q_2 + \gamma_k \otimes \gamma_k^{-1} q_2^2) \right] \cos(\pi \theta_3^k) \prod_{i=1}^2 \left( 2 \sin(\pi \theta_i^k) \right) , \\ \mathcal{M}_{\mathcal{N}=2}^{(k)} & = \left( \frac{B}{4\pi^2} \right)^2 \int_0^{\infty} \frac{dt}{2t} \text{Tr} \left[ -2q^2 \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \right] \cos(\pi R_3^k) \prod_{i=1}^2 \left( 2 \sin(\pi R_i^k) \right) , \end{aligned} \quad (3.11)$$

where the product is over the two twisted angles and  $\theta_3$  (and  $R_3$ ) denote the untwisted direction. These expressions are exact due to the  $\mathcal{N} = 2$  structure. With  $\mathcal{N} = 2$  supersymmetry only BPS multiplets can renormalise the gauge couplings and the string oscillator tower is all non-BPS. As a result in a purely local computation the only non-zero contribution comes from the zero modes.

Evaluated in the IR limit  $t \rightarrow \infty$  the magnetised  $B^2$  amplitudes must reproduce the field theory  $\beta$ -functions. Evaluated in the UV limit  $t \rightarrow 0$  the amplitudes give the threshold corrections to the gauge couplings. As discussed in the introduction the key feature of the  $\mathcal{N} = 2$  sector is that, since the expressions are exact, the running is with the same coefficient in the IR and the UV. Evaluated in a purely local model, such  $\mathcal{N} = 2$  sectors give logarithmic ultraviolet divergences,  $\int \frac{dt}{t} \Delta^{\mathcal{N}=2}$ . This ultraviolet divergence is associated with a tadpole for partially twisted field. In a global model these divergences are cutoff as global tadpole cancellation occurs. From the closed string channel, this corresponds to the existence of new brane/O-plane sectors located in the bulk which also act as sources for the partially twisted field and cancel the tadpole sourced in the local model.

From the open string viewpoint the incorporation of these sectors corresponds to the inclusion of winding modes from the singularity to the distant bulk branes/O-planes. Such modes are charged and BPS and contribute to the threshold corrections, cutting off the  $\beta$ -function running from the  $\mathcal{N} = 2$  sector. The details of the cutoff depend on the precise and model-dependent location of the bulk branes, but what is model-independent is that the winding modes act as an effective ultraviolet cutoff on the  $\mathcal{N} = 2$  sector, cutting off the running at a mass scale  $M_X = RM_s$ .

For the  $\mathcal{N} = 1$  sectors there is no such decoupling. In the IR the  $\mathcal{N} = 1$  sector combines with the  $\mathcal{N} = 2$  contributions to give the field theory  $\beta$  functions. In the UV the string oscillator tower enters giving a non-vanishing contribution. For non-abelian generators the threshold corrections vanish in the far UV as closed string tadpole cancellation is enforced. For  $U(1)$  generators the threshold corrections can diverge due to an on-shell exchange of a  $\mathcal{N} = 1$  twisted RR mode via a Green-Schwarz coupling  $C_2 \wedge \text{Tr} F$ . The abelian case is discussed in detail in appendix A but will not feature in the main text.

Similar to the way winding modes give an effective cutoff at  $M_X = RM_s$  for  $\mathcal{N} = 2$  sectors, the oscillator modes give an effective cutoff at  $M_s$  for  $\mathcal{N} = 1$  sectors. As  $\mathcal{N} = 1$  sectors are purely local tadpole cancellation occurs once the open string oscillators are included. As  $t \ll 1$  gives  $l \sim 1/t \gg 1$ , in this limit all higher closed string modes are exponentially decoupled and the amplitude reduces to the (vanishing) IR closed string tadpole. Modulo small corrections that do not depend on the overall volume, the effective cutoff for the  $\mathcal{N} = 1$  sectors is therefore at  $t = 1/M_s^2$ .

The general amplitude therefore looks like (1.8) which we recall here

$$\frac{1}{g^2}(\mu) = \frac{1}{g^2}\Big|_0 + \beta_a \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_a^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right). \quad (3.12)$$

As this involves different running between  $M_X$  and  $M_s$  and  $M_s$  and  $\mu$  depending on the relative size of  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  contributions to the beta function, in general this appears to differ with the Kaplunovsky-Louis formula (1.1) which only contains field theory running from the winding scale  $M_X$ .

In the next sections we shall study this issue in detail for the  $\mathbb{C}^3/\mathbb{Z}_4$  orientifold, and see how the discrepancy can be resolved.

### 3.2 The $\mathbb{Z}_4$ case

We now specialise the above formulae to the case of the  $\mathbb{Z}_4$  orientifold. We begin by turning on a background field within the  $SU(n_1)$  gauge group. Note that since the orientifold identifies  $SU(n_1)$  and  $SU(n_3)$  we must turn on the background field for both. The normalisation is fixed by the canonical gauge field normalisation  $\text{Tr} q^2 = \frac{1}{2}$ . The charge matrices are then given by

$$q = q_1 = -q_2 = \text{diag}(\mathbf{0}_{n_0}, Q_{SU(n_1)}, \mathbf{0}_{n_2}, Q_{SU(n_1)}) , \quad (3.13)$$

$$Q_{SU(n_1)} = \frac{1}{\sqrt{8}} \text{diag}(1, -1, 0, \dots, 0) . \quad (3.14)$$



Evaluating the amplitudes from section 3.1, summing over the  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sectors separately we find explicitly

$$\begin{aligned}\mathcal{A}_{\mathcal{N}=1}^{SU(n_1)} &= \frac{1}{4} \sum_{k=1,3} \mathcal{A}_{\mathcal{N}=1}^{(k)} = 0 , \\ \mathcal{M}_{\mathcal{N}=1}^{SU(n_1)} &= \frac{1}{4} \sum_{k=1,2,3,4} \mathcal{M}_{\mathcal{N}=1}^{(k)} = 0 ,\end{aligned}\tag{3.15}$$

$$\begin{aligned}\mathcal{A}_{\mathcal{N}=2}^{SU(n_1)} &= \frac{1}{4} \mathcal{A}_{\mathcal{N}=2}^{(2)} = -(2n_1 - 2n_2 - 4) \left( \frac{B}{4\pi^2} \right)^2 \int_0^\infty \frac{dt}{2t} , \\ \mathcal{M}_{\mathcal{N}=2}^{SU(n_1)} &\equiv 0 .\end{aligned}\tag{3.16}$$

We can extract the  $\beta_{SU(n_1)}$  function by imposing IR and UV cutoffs on the  $\mathcal{N} = 2$  integral set by the probe energy scale  $1/\mu^2$  and the winding modes scale  $M_X = RM_s$  respectively. Then using (3.3) we get

$$\begin{aligned}\frac{\Lambda_2^{SU(n_1)}}{8\pi^2} &= -\frac{1}{8\pi^2} (2n_1 - 2n_2 - 4) \int_{\frac{1}{M_X^2}}^{\frac{1}{\mu^2}} \frac{dt}{2t} \\ &= \beta_{SU(n_1)} \ln \left( \frac{M_X^2}{\mu^2} \right) .\end{aligned}\tag{3.17}$$

which exactly matches the expected field theory result using (3.2) and (2.19). This is the same behaviour that was observed in [7] for the purely orbifold case.

We now turn to the  $SO(n_0)$  gauge group and turn on the generator

$$q = q_1 = -q_2 = \text{diag} (Q_{SO(n_0)}, \mathbf{0}_{n_1}, \mathbf{0}_{n_2}, \mathbf{0}_{n_1}) ,\tag{3.18}$$

$$Q_{SO(n_0)} = \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 & \dots \\ -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} .\tag{3.19}$$

Evaluating the amplitudes we find, using the tadpoles (2.17),

$$\begin{aligned}\mathcal{A}_{\mathcal{N}=1}^{SO(n_0)} &= \frac{1}{4} \sum_{k=1,3} \mathcal{A}_{\mathcal{N}=1}^{(k)} \xrightarrow{IR} - \left( \frac{B}{4\pi^2} \right)^2 \int_{t'}^\infty \frac{dt}{2t} 2(n_0 - n_2) = +8 \left( \frac{B}{4\pi^2} \right)^2 \int_{t'}^\infty \frac{dt}{2t} , \\ \mathcal{M}_{\mathcal{N}=1}^{SO(n_0)} &= \frac{1}{4} \sum_{k=1,2,3} \mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow{IR} 6 \left( \frac{B}{4\pi^2} \right)^2 \int_{t'}^\infty \frac{dt}{2t} ,\end{aligned}\tag{3.20}$$

$$\begin{aligned}\mathcal{A}_{\mathcal{N}=2}^{SO(n_0)} &= \frac{1}{4} \mathcal{A}_{\mathcal{N}=2}^{(2)} = -(n_0 + n_2 - 2n_1) \left( \frac{B}{4\pi^2} \right)^2 \int_0^\infty \frac{dt}{2t} = -(2n_0 - 2n_1 + 4) \left( \frac{B}{4\pi^2} \right)^2 \int_0^\infty \frac{dt}{2t} , \\ \mathcal{M}_{\mathcal{N}=2}^{SO(n_0)} &\equiv 0 .\end{aligned}\tag{3.21}$$

As described above tadpole cancellation ensures that in the UV the  $\mathcal{N} = 1$  annulus and Mobius amplitudes cancel against each other. We can check this UV cancellation using the

expressions (3.10) which give

$$\mathcal{A}_{\mathcal{N}=1}^{SO(n_0)} \xrightarrow{\nu \rightarrow \infty} - \left( \frac{B}{4\pi^2} \right)^2 \int_{\nu'}^{\infty} dl \, 4(n_0 - n_2) = \left( \frac{B}{4\pi^2} \right)^2 \int_{\nu'}^{\infty} dl \, 16 , \quad (3.22)$$

$$\mathcal{M}_{\mathcal{N}=1}^{SO(n_0)} \xrightarrow{\nu \rightarrow \infty} - \left( \frac{B}{4\pi^2} \right)^2 \int_{\nu'}^{\infty} dl \, 16 , \quad (3.23)$$

leading to  $\Lambda_2^{SO(n_0)} \xrightarrow{UV} 0$ . From closed string channel the non-cancelling subleading terms in (3.22) and (3.23) are of order  $e^{-\pi l/M_s^2}$  and so vanish exponentially once  $l \gg M_s^2$  or  $t \ll 1/M_s^2$ . Therefore, up to small additional corrections we obtain an effective cutoff at  $M_s$  for the  $\mathcal{N} = 1$  amplitudes and an effective cutoff at  $M_X$  for  $\mathcal{N} = 2$  amplitudes. To compare with the Kaplunovsky-Louis expression we impose these cutoffs on the  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sector, also taking the IR cutoff at  $1/\mu^2$ . We find

$$\begin{aligned} \frac{\Lambda_2^{SO(n_0)}}{8\pi^2} &= \frac{1}{16\pi^2} (-14) \ln \left( \frac{\mu^2}{M_s^2} \right) + \frac{1}{16\pi^2} (2n_0 - 2n_1 + 4) \ln \left( \frac{\mu^2}{M_X^2} \right) \\ &= \frac{1}{16\pi^2} (2n_0 - 2n_1 - 10) \ln \left( \frac{\mu^2}{M_s^2} \right) + \frac{1}{16\pi^2} (2n_0 - 2n_1 + 4) \ln \left( \frac{M_s^2}{M_X^2} \right) \\ &\equiv \left( \beta_{SO(n_0)}^{\mathcal{N}=1} + \beta_{SO(n_0)}^{\mathcal{N}=2} \right) \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_{SO(n_0)}^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right) \end{aligned} \quad (3.24)$$

$$= \beta_{SO(n_0)} \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_{SO(n_0)}^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right) , \quad (3.25)$$

where we defined  $\beta_{SO(n_0)}^{\mathcal{N}=1}$  and  $\beta_{SO(n_0)}^{\mathcal{N}=2}$  as the contributions in the IR to the  $\beta$  functions coming from the  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  sectors respectively. The expression (3.24) differs from the naive application of the KL formula (1.2). The difference arising from a non-vanishing contribution to the  $\beta$  function from the  $\mathcal{N} = 1$  sector.

To study the  $Sp(n_2)$  gauge group we turn on the generator

$$q = q_1 = -q_2 = \text{diag}(\mathbf{0}_{n_0}, \mathbf{0}_{n_1}, Q_{Sp(n_2)}, \mathbf{0}_{n_1}) , \quad (3.26)$$

$$Q_{Sp(n_2)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} . \quad (3.27)$$

In a similar fashion this gives the result

$$\frac{\Lambda_2^{Sp(n_2)}}{8\pi^2} = \left( \beta_{Sp(n_2)}^{\mathcal{N}=1} + \beta_{Sp(n_2)}^{\mathcal{N}=2} \right) \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_{Sp(n_2)}^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right) , \quad (3.28)$$

where

$$\beta_{Sp(n_2)}^{\mathcal{N}=1} = \frac{1}{16\pi^2} (14) = -\beta_{SO(n_0)}^{\mathcal{N}=1} , \quad (3.29)$$

$$\beta_{Sp(n_2)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (2n_0 - 2n_1 + 4) = -\beta_{SO(n_0)}^{\mathcal{N}=2} . \quad (3.30)$$

The physics is therefore the same as the  $SO(n_0)$  case: the effective  $\beta$  function undergoes a jump at  $M_s$  and so there are two distinct phases of the running, one from  $M_X$  to  $M_s$  and one from  $M_s$  to  $\mu$ .

Gathering these results together, we have

$$\begin{aligned}\frac{\Lambda_2^{SO(n_0)}}{8\pi^2} &= \left(\Delta + \beta_{SO(n_0)}^{\mathcal{N}=2}\right) \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{SO(n_0)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right), \\ \frac{\Lambda_2^{SU(n_1)}}{8\pi^2} &= \beta_{SU(n_1)}^{\mathcal{N}=2} \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{SU(n_1)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right), \\ \frac{\Lambda_2^{Sp(n_2)}}{8\pi^2} &= \left(-\Delta + \beta_{Sp(n_2)}^{\mathcal{N}=2}\right) \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{Sp(n_2)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right),\end{aligned}\tag{3.31}$$

where  $\Delta = \beta_{SO(n_0)}^{\mathcal{N}=1} = -\beta_{Sp(n_2)}^{\mathcal{N}=1}$ .

The form of the gauge couplings differ from the naive application of the KL formula as in (1.2). The difference lies in the presence of the  $\Delta$  term in equations (3.31) associated to  $\mathcal{N} = 1$  sectors which contribute to running from the string scale but not from the winding scale. We now proceed to study how this discrepancy is resolved.

#### 4. Threshold corrections: matching the field theory

Recall that the string calculation gives the coefficient multiplying the  $F^2$  term in the Lagrangian which in the field theory also includes a tree-level coupling to chiral superfields  $M_k$

$$\mathcal{L} = \left(\frac{1}{g_a^2} + \sum_{i=1}^K s_k^a \text{Re} M_k\right) \text{Tr} F_a^2.\tag{4.1}$$

The coupling (4.1) comes from the holomorphic gauge kinetic function

$$f_a(\Phi) = S + s_k^a M_k.\tag{4.2}$$

The chiral superfields  $M_k$  correspond to closed string twisted modes with  $\text{Re} M_k$  corresponding to the NS-NS part and  $\text{Im} M_k$  the RR part.

Geometrically the twisted modes correspond to collapsed two- and four-cycles. They can therefore be thought of as dimensionally reducing the NS and RR supergravity form fields on the collapsed two-cycle. These fields descend from reducing either  $J \wedge J + iC_4$  or  $B_2 + iC_2$  on collapsed two/four-cycles.<sup>6</sup> Here  $J$  is the Kahler form,  $B_2$  the NS two-form, and  $C_2$  and  $C_4$  are the RR two and four-form respectively. Depending on whether we reduce the above fields on cycles or their dual cycles we can obtain either linear or chiral multiplets depending on whether the bosonic 4d fields associated to the reduction of  $C_4$  are

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<sup>6</sup>The NS two-form  $B_2$  splits into a part that is even and a part that is odd under the orientifold action  $B_2 = B_2^+ + B_2^-$ .  $B_2^-$  parameterises the modulus (by abuse of notation we label this  $B_2$  in the main text) and therefore vanishes (at tree level) at the singularity. The field  $B_2^+$  can have a non-vanishing vev at the singularity as in [23, 24].

scalars or 2-forms. We denote the chiral multiplet by  $M_k$  and the real scalar component of the linear multiplet by  $m_k$ .

In [6] the supergravity analysis that led to the KL formula (1.1) is carried out using chiral multiplets. Therefore to compare the string result with the supergravity formula we need to dualise the linear multiplet to a chiral multiplet. This procedure is described in detail in [18] for the heterotic string. In appendix D we review this and also discuss the IIB case. Performing a similar analysis for the local IIB models the result is that at tree level<sup>7</sup>

$$\text{Re}M_k = m_k , \quad (4.3)$$

with  $m_k$  the linear multiplet and  $M_k$  the chiral multiplet.  $m_k$  always vanishes at the singularity where  $M_k$  only vanishes at tree-level. However at 1-loop level this is modified to (1.9) which we reproduce here

$$\text{Re}(M_k) = m_k - \alpha_k \ln R^2 . \quad (4.4)$$

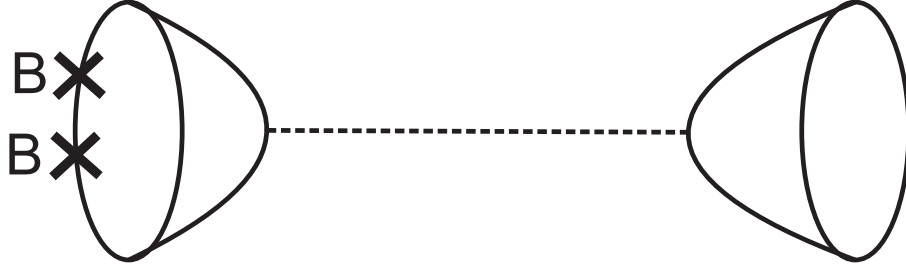
An important point is that the 1-loop field redefinition (4.4) is tied to the 1-loop correction to the gauge kinetic function and is therefore only present when the particular  $\mathcal{N} = 1$  twisted mode  $m_k$  contributes to the  $\beta$  functions. As discussed in the introduction, this precisely reproduces the behaviour required to match the string and field theory results if the coupling  $s_k^a$  are proportional to the  $\beta_a^{(k)}$ . In the next section we explicitly perform the string calculation to check this proportionality.

#### 4.1 Extracting closed string couplings

In order to calculate the correction to the gauge couplings induced by the field redefinitions we need to know the coupling  $s_k^a$  of the twisted closed string modes to the gauge field strengths  $F_{\mu\nu}^a F^{a,\mu\nu}$ . One way to calculate this is by extracting the UV divergence of the  $B^4$  amplitude which corresponds to exchanging an on-shell twisted modes sourced by the magnetic field background [17]. However this method only gives  $(s_{ak})^2$  and so is insensitive to the sign of  $s_{ak}$  which for us plays a crucial role. The method we employ is to study the open string UV divergence of the  $B^2$  Annulus amplitudes. In the closed string tree level channel this amplitude can be interpreted as a vertex between the NSNS closed string twisted mode and the gauge field strength, sourcing the twisted mode which is then absorbed by the vacuum, see figure 4.1. Of course the overall diagram vanishes once tadpole cancellation is imposed but the  $(\text{Re}M_k) F_{\mu\nu} F^{\mu\nu}$  coupling can be extracted by stripping off the tadpole piece which just corresponds to the trace over the other end of the string. This is because the tadpoles give this coupling for the RR fields but due to supersymmetry these are equivalent up to a constant to the NS tadpoles.<sup>8</sup>

<sup>7</sup>The analysis is basically the same as the heterotic case [18] but with the dilaton replaced by the twisted mode. The only significant change is that the Kahler potential for the twisted mode is quadratic  $K \sim m_k^2$  rather than logarithmic.

<sup>8</sup>Each twisted sector gives rise to two real closed string modes and their coupling is given by the real and imaginary parts of  $\text{Tr}\gamma_\theta^k$ . However in our orientifolds the imaginary part will always vanish corresponding to projecting out that twisted mode.



**Figure 1:** From a supergravity perspective this represents an  $(\text{Re}M_k) F_{\mu\nu} F^{\mu\nu}$  vertex sourcing an  $M_k$  field which propagates and is then absorbed by the vacuum tadpole. By factoring out the vacuum tadpole we can infer the coefficient of the  $(\text{Re}M_k) F_{\mu\nu} F^{\mu\nu}$  coupling.

## 4.2 The $\mathbb{Z}_4$ case

In this section we calculate the  $\mathcal{N} = 1$  twisted mode coupling to the gauge fields for the  $\mathbb{Z}_4$  case and show that it takes the form appropriate for reconciling the string calculation of section 3 with the field theory KL formula (1.1).

The Annulus UV amplitudes read

$$\mathcal{A}_{\mathcal{N}=1}^{SO(n_0)} \xrightarrow[l' \rightarrow \infty]{UV} - \left( \frac{B}{4\pi^2} \right)^2 \int_{l'}^{\infty} dl 4(n_0 - n_2) , \quad (4.5)$$

$$\mathcal{A}_{\mathcal{N}=1}^{Sp(n_2)} \xrightarrow[l' \rightarrow \infty]{UV} - \left( \frac{B}{4\pi^2} \right)^2 \int_{l'}^{\infty} dl 4(n_2 - n_0) , \quad (4.6)$$

$$\mathcal{A}_{\mathcal{N}=1}^{SU(n_1)} \xrightarrow[l' \rightarrow \infty]{UV} 0 . \quad (4.7)$$

There is a single closed string twisted mode  $m_0$  and its coupling to the vacuum is given by  $\text{Tr}\gamma_\theta = (n_0 - n_2)$  which gives

$$s_0^{SO(n_0)} = \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_0} , \quad s_0^{Sp(n_2)} = \frac{\beta_{Sp(n_2)}^{\mathcal{N}=1}}{\alpha_0} , \quad s_0^{SU(n_1)} = 0 . \quad (4.8)$$

Here  $\alpha_0$  is some (gauge group) universal constant that corresponds to extracting the appropriately normalised coupling and propagator. We therefore find the required result that the coupling are proportional to the  $\mathcal{N} = 1$   $\beta$  functions. Recall that in terms of the gauge kinetic functions this reads

$$f_{SO(n_0)} = S + \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_0} M_0 , \quad (4.9)$$

$$f_{SU(n_1)} = S , \quad (4.10)$$

$$f_{Sp(n_2)} = S + \frac{\beta_{Sp(n_2)}^{\mathcal{N}=1}}{\alpha_0} M_0 . \quad (4.11)$$

We therefore see that if at the orbifold point the chiral superfield

$$\text{Re}M_0 = -\alpha_0 \ln R^2 , \quad (4.12)$$

the holomorphic gauge couplings become non-universal. The string results then match exactly the field theory formula with

$$s_0^a \text{Re} M_0 = -\beta_a^{\mathcal{N}=1} \ln \left( \frac{M_X^2}{M_s^2} \right). \quad (4.13)$$

The striking point is that a single field redefinition is capable of altering three  $\beta$  functions in a way to resolve the discrepancy with the naive use of the Kaplunovsky-Louis formula.

## 5. More examples

In this section we present two more examples of orientifolded singularities,  $\mathbb{Z}_6$  and  $\mathbb{Z}_6'$ , that serve as checks on the above analysis and understanding. As with the  $\mathbb{Z}_4$  case, as far as we are aware these have not been previously presented in the literature and so we outline their construction before moving on to the magnetised amplitude calculations. These orientifolds exhibit more structure compared to the  $\mathbb{Z}_4$  example through the presence of more  $\mathcal{N} = 1/\mathcal{N} = 2$  twisted closed string modes.

### 5.1 The $\mathbb{C}^3/\mathbb{Z}_6$ orientifold

The  $\mathbb{Z}_6$  orbifold action is generated by  $\theta = (1/6, 1/6, -1/3)$ . We take the orientifold spatial action to be  $R = (7/12, 1/12, -2/3)$ . The orientifold group is therefore

$$\left\{ (0, 0, 0), \left( \frac{1}{6}, \frac{1}{6}, -\frac{1}{3} \right), \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right), \left( \frac{1}{2}, \frac{1}{2}, -1 \right), \left( \frac{2}{3}, \frac{2}{3}, -\frac{4}{3} \right), \left( \frac{5}{6}, \frac{5}{6}, -\frac{5}{3} \right), \right. \\ \Omega I \left( \frac{7}{12}, \frac{1}{12}, -\frac{2}{3} \right), \Omega I \left( \frac{3}{4}, \frac{1}{4}, -1 \right), \Omega I \left( \frac{11}{12}, \frac{5}{12}, -\frac{4}{3} \right), \Omega I \left( \frac{13}{12}, \frac{7}{12}, -\frac{5}{3} \right), \\ \left. \Omega I \left( \frac{5}{4}, \frac{3}{4}, -2 \right), \Omega I \left( \frac{17}{12}, \frac{11}{12}, -\frac{7}{3} \right) \right\}, \quad (5.1)$$

Including the spatial action of  $I$  the fixed point locus consists solely of the origin. We take the orbifold generating element

$$\gamma_\theta = \text{diag}(\mathbf{1}_{n_0}, \alpha \mathbf{1}_{n_1}, \alpha^2 \mathbf{1}_{n_2}, \alpha^3 \mathbf{1}_{n_3}, \alpha^4 \mathbf{1}_{n_4}, \alpha^5 \mathbf{1}_{n_5}) \quad \text{with } \alpha = e^{\pi i/3}, \quad (5.2)$$

and impose  $n_1 = n_5$  and  $n_2 = n_4$ . For the orientifold action we take

$$\gamma_{\Omega'} = \begin{pmatrix} \mathbf{1}_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{1/2} \epsilon_{n_5} \\ 0 & 0 & 0 & 0 & \alpha \epsilon_{n_4} & 0 \\ 0 & 0 & 0 & \alpha^{3/2} \epsilon_{n_3} & 0 & 0 \\ 0 & 0 & \alpha^2 \epsilon_{n_2} & 0 & 0 & 0 \\ 0 & \alpha^{5/2} \epsilon_{n_1} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.3)$$

with  $\epsilon$  denoting the anti-symmetric matrix with unit off-diagonal entries.

Multiplicity	Representation			
	$SO(n_0)$	$SU(n_1)$	$SU(n_2)$	$Sp(n_3)$
2	$n_0$	$\bar{n}_1$	1	1
2	1	$n_1$	$\bar{n}_2$	1
2	1	1	$n_2$	$n_3$
1	$n_0$	1	$n_2$	1
1	1	$\bar{n}_1$	1	$n_3$
1	1	1	$\bar{A}_{n_2}$	1
1	1	$S_{n_1}$	1	1

**Table 2:** Field content and representations for  $Z_6$  orientifold. The  $n_i$  denote the fundamental representation and  $S$  and  $A$  denote symmetric and anti-symmetric representations respectively.

Calculating the tadpoles using (2.10-2.11) leads to

$$\text{Tr} [\gamma_\theta] - 8 = \text{Tr} [\gamma_{\theta^2}] = 0 . \quad (5.4)$$

There are two real closed string  $\mathcal{N} = 1$  twisted modes  $m_0$  and  $m_1$  associated with the first and second tadpole conditions in (5.4) respectively. The tadpole constraints (5.4) impose the conditions

$$\begin{aligned} n_2 &= n_0 - 4 , \\ n_3 &= n_1 - 4 . \end{aligned} \quad (5.5)$$

The massless fermionic spectrum of the theory can be calculated using (2.9) which gives the matter content shown in table 2. The gauge group is

$$G = SO(n_0) \times U(n_1) \times U(n_2) \times Sp(n_3) . \quad (5.6)$$

The non-abelian anomalies of the theory correspond to (5.5). We are also interested in the beta functions for the gauge groups and these read, after imposing anomaly cancellation (5.5),

$$\beta_{SO(n_0)} = \frac{1}{16\pi^2} (-2n_0 + 2n_1 + 2) = -\beta_{Sp(n_3)} , \quad (5.7)$$

$$\beta_{SU(n_1)} = \frac{1}{16\pi^2} (2n_0 - 2n_1 - 5) = -\beta_{SU(n_2)} . \quad (5.8)$$

The calculation for the threshold corrections proceeds as in section 3.2 and here we quote the relevant results. The vacuum energies read

$$\frac{\Lambda_2^a}{8\pi^2} = (\beta_a^{\mathcal{N}=1} + \beta_a^{\mathcal{N}=2}) \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_a^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right) , \quad (5.9)$$

where we have

$$\beta_{SO(n_0)}^{\mathcal{N}=1} = -\beta_{Sp(n_3)}^{\mathcal{N}=1} = \frac{1}{16\pi^2} (-2) , \quad \beta_{SO(n_0)}^{\mathcal{N}=2} = -\beta_{Sp(n_3)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (-2n_0 + 2n_1 + 4) \quad (5.10)$$

$$\beta_{SU(n_1)}^{\mathcal{N}=1} = -\beta_{SU(n_2)}^{\mathcal{N}=1} = -\frac{1}{16\pi^2} , \quad \beta_{SU(n_1)}^{\mathcal{N}=2} = -\beta_{SU(n_2)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (2n_0 - 2n_1 - 4) . \quad (5.11)$$

An important point is that the  $\mathcal{N} = 1$  contributions to  $\Lambda_2^a$  comes solely from the  $\gamma_\theta$  sector. In terms of the  $\mathcal{N} = 1$  twisted modes  $m_0$  and  $m_1$  we have (as in section 1)

$$\begin{aligned}\beta_a^0 &= \beta_a^{\mathcal{N}=1} , \\ \beta_a^1 &= 0 .\end{aligned}\tag{5.12}$$

The gauge couplings take the form

$$\begin{aligned}\frac{\Lambda_2^{SO(n_0)}}{8\pi^2} &= \left(-2\Delta + \beta_{SO(n_0)}^{\mathcal{N}=2}\right) \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{SO(n_0)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right) , \\ \frac{\Lambda_2^{SU(n_1)}}{8\pi^2} &= \left(-\Delta + \beta_{SU(n_1)}^{\mathcal{N}=2}\right) \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{SU(n_1)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right) , \\ \frac{\Lambda_2^{SU(n_2)}}{8\pi^2} &= \left(\Delta + \beta_{SU(n_2)}^{\mathcal{N}=2}\right) \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{SU(n_2)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right) , \\ \frac{\Lambda_2^{Sp(n_3)}}{8\pi^2} &= \left(2\Delta + \beta_{Sp(n_3)}^{\mathcal{N}=2}\right) \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_{Sp(n_3)}^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right) ,\end{aligned}\tag{5.13}$$

where as before  $\Delta = \beta_{SO(n_0)}^{\mathcal{N}=2}$ .

We can now check that the closed string twisted modes couple in the correct way to match the string calculation with the field theory results. In this case we have two closed string twisted modes  $m_0$  and  $m_1$ . We find

$$\begin{aligned}s_0^{SO(n_0)} &= -s_0^{Sp(n_3)} = \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_0} , \quad s_1^{SO(n_0)} = s_1^{Sp(n_3)} = \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_1} , \\ s_0^{SU(n_1)} &= -s_0^{SU(n_2)} = \frac{\beta_{SU(n_1)}^{\mathcal{N}=1}}{\alpha_0} , \quad s_1^{SU(n_1)} = s_1^{SU(n_2)} = -\frac{\beta_{SU(n_1)}^{\mathcal{N}=1}}{\alpha_1} .\end{aligned}\tag{5.14}$$

Here  $\alpha_0$  and  $\alpha_1$  are constants (different from their  $\mathbb{Z}_4$  values). In terms of the gauge kinetic functions this reads

$$\begin{aligned}f_{SO(n_0)} &= S + \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_0} M_0 + \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_1} M_1 , \\ f_{SU(n_1)} &= S + \frac{\beta_{SU(n_1)}^{\mathcal{N}=1}}{\alpha_0} M_0 - \frac{\beta_{SU(n_1)}^{\mathcal{N}=1}}{\alpha_1} M_1 , \\ f_{SU(n_2)} &= S + \frac{\beta_{SU(n_2)}^{\mathcal{N}=1}}{\alpha_0} M_0 + \frac{\beta_{SU(n_2)}^{\mathcal{N}=1}}{\alpha_1} M_1 , \\ f_{Sp(n_3)} &= S + \frac{\beta_{Sp(n_3)}^{\mathcal{N}=1}}{\alpha_0} M_0 - \frac{\beta_{Sp(n_3)}^{\mathcal{N}=1}}{\alpha_1} M_1 .\end{aligned}\tag{5.15}$$

There is new structure compared to the  $\mathbb{Z}_4$  case. We now have two linear multiplet  $\mathcal{N} = 1$  twisted modes  $m_0$  and  $m_1$ .  $m_0$  has an appropriate coupling proportional to the  $\beta$  functions, but  $m_1$  does not as its coupling to the  $SU(n_1)$  and  $Sp(n_3)$  gauge groups has the wrong sign. Therefore we expect that in this case  $M_0$  undergoes a one-loop redefinition restoring consistency with the Kaplunovsky-Louis formula while  $M_1$  does not undergo a



redefinition and still has vanishing vev at the singularity. This matches the fact that the 1-loop redefinition is proportional to the contribution to the  $\beta$  functions and the result (5.12).

From the string perspective the redefinition is related to the fact that fractional O-planes wrap the  $m_1$  collapsed cycle and contribute to the  $m_1$  tadpole, while O-planes do not wrap the  $m_2$  cycle and in closed string channel this tadpole is wholly sourced by annulus diagrams.

## 5.2 The $\mathbb{C}^3/\mathbb{Z}'_6$ orientifold

The  $\mathbb{Z}'_6$  orbifold action is generated by  $\theta = (1/6, 1/3, -1/2)$ . We take the orientifold spatial action to be  $R = (7/12, -1/3, -1/4)$ . The orientifold group is therefore

$$\begin{aligned} & \left\{ (0, 0, 0), \left(\frac{1}{6}, \frac{1}{3}, -\frac{1}{2}\right), \left(\frac{1}{3}, \frac{2}{3}, -1\right), \left(\frac{1}{2}, 1, -\frac{3}{2}\right), \left(\frac{2}{3}, \frac{4}{3}, -2\right), \left(\frac{5}{6}, \frac{5}{3}, -\frac{5}{2}\right), \right. \\ & \Omega I \left(\frac{7}{12}, -\frac{1}{3}, -\frac{1}{4}\right), \Omega I \left(\frac{3}{4}, 0, -\frac{3}{4}\right), \Omega I \left(\frac{11}{12}, \frac{1}{3}, -\frac{5}{4}\right), \Omega I \left(\frac{13}{12}, \frac{2}{3}, -\frac{7}{4}\right), \\ & \left. \Omega I \left(\frac{5}{4}, 1, -\frac{9}{4}\right), \Omega I \left(\frac{17}{12}, \frac{4}{3}, -\frac{11}{4}\right) \right\}, \end{aligned} \quad (5.16)$$

The tadpoles are

$$\text{Tr}[\gamma_\theta] + 4 = 0. \quad (5.17)$$

This is associated with a single  $\mathcal{N} = 1$  twisted closed string mode. The CP embedding is the same as for  $\mathbb{Z}_6$  but now the tadpole constraint reads

$$-n_0 - n_1 + n_2 + n_3 - 4 = 0. \quad (5.18)$$

The massless fermionic spectrum gives the matter content shown in table 3 and the gauge group is the same as  $\mathbb{Z}_6$ . The non-abelian anomalies match the tadpoles (5.18). Using the tadpoles (5.18) to eliminate  $n_3$ , the  $\beta$  functions read

$$\beta_{SO(n_0)} = \frac{1}{16\pi^2} (-2n_0 + 2n_1 + 10), \quad (5.19)$$

$$\beta_{SU(n_1)} = \frac{1}{16\pi^2} (n_0 - 2n_1 + n_2 + 3), \quad (5.20)$$

$$\beta_{SU(n_2)} = \frac{1}{16\pi^2} (n_0 + 2n_1 - 3n_2 + 1), \quad (5.21)$$

$$\beta_{Sp(n_3)} = \frac{1}{16\pi^2} (-2n_0 - 2n_1 + 4n_2 - 18). \quad (5.22)$$

The calculation for the threshold corrections gives the vacuum energies

$$\frac{\Lambda_2^a}{8\pi^2} = (\beta_a^{\mathcal{N}=1} + \beta_a^{\mathcal{N}=2}) \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_a^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right), \quad (5.23)$$

Multiplicity	Representation			
	$SO(n_0)$	$SU(n_1)$	$SU(n_2)$	$Sp(n_3)$
1	$n_0$	$\bar{n}_1$	1	1
1	1	$n_1$	$\bar{n}_2$	1
1	1	1	$n_2$	$n_3$
1	$n_0$	1	$\bar{n}_2$	1
1	1	$n_1$	1	$n_3$
1	$n_0$	1	1	$n_3$
1	1	$n_1$	$n_2$	1
1	1	$\bar{n}_1$	$\bar{n}_2$	1
1	1	1	$A_{n_2}$	1
1	1	$\bar{S}_{n_1}$	1	1

**Table 3:** Field content and representations for  $\mathbb{Z}'_6$  orientifold. The  $n_i$  denote the fundamental representation and  $S$  and  $A$  denote symmetric and anti-symmetric representations respectively.

where we have

$$\beta_{SO(n_0)}^{\mathcal{N}=1} = -\beta_{Sp(n_3)}^{\mathcal{N}=1} = \frac{1}{16\pi^2} (10) , \quad (5.24)$$

$$\beta_{SU(n_1)}^{\mathcal{N}=1} = -\beta_{SU(n_2)}^{\mathcal{N}=1} = \frac{1}{16\pi^2} (5) , \quad (5.25)$$

$$\beta_{SO(n_0)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (-2n_0 + 2n_1) , \quad (5.26)$$

$$\beta_{SU(n_1)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (n_0 - 2n_1 + n_2 - 2) , \quad (5.27)$$

$$\beta_{SU(n_2)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (n_0 + 2n_1 - 3n_2 + 6) , \quad (5.28)$$

$$\beta_{Sp(n_3)}^{\mathcal{N}=2} = \frac{1}{16\pi^2} (-2n_0 - 2n_1 + 4n_2 - 8) . \quad (5.29)$$

We can now check that the closed string twisted modes couple in the correct way to match the string calculation with the field theory results. In this case we have one closed string twisted mode  $m_0$ . We find

$$\begin{aligned} s_0^{SO(n_0)} &= -s_0^{Sp(n_3)} = \frac{\beta_{SO(n_0)}^{\mathcal{N}=1}}{\alpha_0} , \\ s_0^{SU(n_1)} &= -s_0^{SU(n_2)} = \frac{\beta_{SU(n_1)}^{\mathcal{N}=1}}{\alpha_0} . \end{aligned} \quad (5.30)$$

We see again that the closed string mode  $m_0$  has the appropriate coupling to match the field theory results after the appropriate redefinition.

### 5.3 D3-D7 Orbifolds

We can also resolve here a puzzle encountered in [7]. In that paper systems of D3/D7 branes on the  $\mathbb{C}^3/\mathbb{Z}_3$  orbifold singularity were considered. As described in [2] such systems

give phenomenologically promising spectra. It was found that the string computation for threshold corrections to the D3 gauge couplings gave universal running between the string and winding scale and non-universal running below the string scale.

We can resolve this issue and find that full agreement with Kaplunovsky-Louis can be found through a redefinition of the two twisted moduli. We briefly summarise the results but for full details of the models refer to [2, 7]. The holomorphic gauge couplings are

$$\begin{aligned} f_{SU(n_0)} &= S + M_1, \\ f_{SU(n_1)} &= S - \frac{M_1}{2} + \frac{\sqrt{3}}{2}M_2, \\ f_{SU(n_2)} &= S - \frac{M_1}{2} - \frac{\sqrt{3}}{2}M_2. \end{aligned} \tag{5.31}$$

The  $\beta$  functions can be written as

$$\frac{\Lambda_2^a}{8\pi^2} = (\beta_a^{\mathcal{N}=1} + \beta_a^{\mathcal{N}=2}) \ln \left( \frac{M_s^2}{\mu^2} \right) + \beta_a^{\mathcal{N}=2} \ln \left( \frac{M_X^2}{M_s^2} \right), \tag{5.32}$$

with

$$\begin{aligned} \beta_{SU(n_0)}^{\mathcal{N}=1} &= \frac{1}{16\pi^2} \left( \frac{2n_0^7 - n_1^7 - n_2^7}{3} \right), \\ \beta_{SU(n_1)}^{\mathcal{N}=1} &= \frac{1}{16\pi^2} \left( \frac{-n_0^7 + 2n_1^7 - n_2^7}{3} \right), \\ \beta_{SU(n_2)}^{\mathcal{N}=1} &= \frac{1}{16\pi^2} \left( \frac{-n_0^7 - n_1^7 + 2n_2^7}{3} \right). \end{aligned} \tag{5.33}$$

Through the redefinitions

$$\text{Re}M_1 = m_1 + \frac{1}{16\pi^2} \left( \frac{2n_0^7 - n_1^7 - n_2^7}{3} \right) \ln R^2, \tag{5.34}$$

$$\frac{\sqrt{3}}{2} \text{Re}M_2 = \frac{\sqrt{3}}{2} m_2 + \frac{1}{16\pi^2} \left( \frac{n_1^7 - n_2^7}{2} \right) \ln R^2, \tag{5.35}$$

we can obtain a full match with Kaplunovsky-Louis. In this case the non-trivial aspect of the redefinition is that two field redefinitions are sufficient to match three gauge couplings. Note that in this case there are two fields  $M_1$  and  $M_2$  contributing from the same twisted sector which did not occur in the orientifold models. This slightly modifies the scenario so that the relationship between the twisted mode gauge coupling  $s_a^k$  and the  $\beta_a$  functions is generalised.

## 6. Conclusions

In this work we studied threshold corrections to the gauge couplings in local models of branes at orientifold singularities. This extends the work of [7] on threshold corrections at orbifold singularities and provides new tractable examples of local models with full CFT

control. For local models the general supergravity analysis performed by Kaplunovsky and Louis [5,6] points towards a unification scale that is enhanced by the bulk radius from the string scale, with field theory running below this winding mode scale.

Our aim has been to understand this formula and address the question of when running starts at the winding mode scale and when running starts at the string scale. We analysed this issue using explicit string calculations and showed that in general this apparent unification at an enhanced scale is only present in particular constructions. The low-energy gauge couplings take the form

$$\frac{1}{g^2}(\mu) = \frac{1}{g^2}\Big|_0 + \beta_a \ln\left(\frac{M_s^2}{\mu^2}\right) + \beta_a^{\mathcal{N}=2} \ln\left(\frac{M_X^2}{M_s^2}\right), \quad (6.1)$$

where  $M_X$  is the winding mode scale. Unification of the gauge coupling at the enhanced scale  $M_X$  occurs whenever there are no non-universal  $\mathcal{N} = 1$  fully twisted, that is modes confined to the singularity, contributions to the gauge coupling threshold corrections. Such modes give contributions to gauge coupling running that start at  $M_s$ . If there is no  $\mathcal{N} = 1$  contribution then the apparent unification at the enhanced scale can be understood [7] from the fact that the remaining  $\mathcal{N} = 2$  contribution gives field theory running up to the winding mode scale where cancellation of global tadpoles implies that the winding modes cut the running off.

We performed detailed calculations to show how this can be reconciled with the Kaplunovsky-Louis formula by an appropriate field redefinition of the  $\mathcal{N} = 1$  closed string modes. This arises from dualising the string linear multiplets to the supergravity chiral multiplets used in the Kaplunovsky-Louis analysis. This required particular couplings of these modes to the gauge fields and the string calculations showed that these couplings are indeed as required.

Understanding the form of  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  contributions to the gauge coupling threshold corrections is therefore important in understanding gauge coupling unification in local models. Open-closed string duality relates the threshold corrections to tadpoles and for orbifold/orientifold models gives a relatively simple rule as to when this occurs: if there is a contribution from multiple diagrams (that transform differently from the IR to the UV) to the local tadpoles then  $\mathcal{N} = 1$  sectors contribute threshold corrections to the gauge couplings. In the examples we presented the D3-D3 cylinder was supplemented by contributions from the Mobius strip in case of orientifolds or the D3-D7 cylinder in the case of D7 branes present.

The fact that generic branes at singularities local models only exhibit gauge coupling unification up to corrections logarithmic in the bulk radius can be attributed to the fact that they are not GUT models: the different gauge groups originate from different branes and the tree-level unification of gauge couplings at the singularity is accidental and does not survive at 1-loop. This comes from the fact that in general twisted sectors couple non-universally to the gauge groups. Although naively it appears that holomorphic gauge couplings are universal at the singularity, we have seen that at one loop level this is not the case, and the non-GUT nature of the setup becomes apparent.

This implies that for true local GUTs this will not occur. We will discuss threshold corrections in this case in [8]. It would also be interesting to study the mirror type IIA picture with intersecting D6 branes. There the Kaplunovsky-Louis formula would again imply that there can be a significant gap between the apparent unification scale and the string scale, especially within the weakly-coupled models of [25]. However there is no geometric picture of a local model and so it would be interesting to understand the relevant string physics. Finally it is clear that questions regarding gauge coupling in local models can only be fully answered once the dynamics of the blow-up fields are understood.

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## A. Anomalous $U(1)$ s

In this section we study anomalous  $U(1)$ s in branes at singularity models. More precisely we study the Green-Schwarz mechanism for generating  $U(1)$  masses, which automatically operates in the case of anomalous  $U(1)$ s but can also affect non-anomalous  $U(1)$ s. This topic does not quite fall within the narrative of the main text and is therefore relegated to the appendix. Further, the cancellation of  $U(1)$  anomalies for models of branes at singularities has been studied following the initial work of [20]. However, the physics of anomalous  $U(1)$ s has some overlap with the gauge threshold corrections discussed in the main text. We also present our analysis using the background field method which differs from the analysis of [20]. We therefore present this appendix as containing partially new results but primarily to illuminate the physics associated to  $U(1)$ s in local models.

The physics of interest is of course the Green-Schwarz mechanism. The relevant terms in the four dimensional theory are  $C_0 F \wedge F$  and  $C_2 \wedge F$ .<sup>9</sup> We can extract these terms in the action using the background field method. The (open string UV, closed string IR) divergence for the  $B^2$  and  $B^4$  amplitudes correspond to the on-shell exchange of the RR mode  $C_2$  and the NS partner of the  $C_0$  field respectively. By turning on a background field for the most general combination of  $U(1)$ s we can extract which  $U(1)$  combinations have each type of coupling. This determines which  $U(1)$ s participate in Green-Schwarz anomaly cancellation and/or become massive.

For purposes of simplicity and clarity we perform the calculations within a  $\mathbb{Z}_4$  orbifold setting (with no orientifolds). The  $\mathbb{Z}_4$  orbifold action is generated by  $\theta = (1/4, 1/4, -1/2)$ . Since there are no orientifolds the tadpoles are sourced purely by the Annulus diagram and therefore read

$$\text{Tr} [\gamma_\theta] = 0 . \quad (\text{A.2})$$

We take the orbifold generating element

$$\gamma_\theta = \text{diag}(\mathbf{1}_{n_0}, \alpha \mathbf{1}_{n_1}, \alpha^2 \mathbf{1}_{n_2}, \alpha^3 \mathbf{1}_{n_3}) \quad \text{with } \alpha = e^{\pi i/2} , \quad (\text{A.3})$$

The tadpole constraints impose the condition

$$\begin{aligned} n_0 &= n_2 , \\ n_1 &= n_3 . \end{aligned} \quad (\text{A.4})$$

---

<sup>9</sup>It is possible to think of these as geometrically arising from say a D7 brane with the Chern-Simons term reduced on a collapsing four-cycle  $\Sigma$  with two-cycle submanifolds  $\omega_\alpha$ . We can write

$$\int_{D7} C_4 \wedge F \wedge F = \int_\Sigma C_4 \int_{M_4} F \wedge F + \int_\Sigma \omega_\alpha \wedge f \int_{M_4} C_2^\alpha \wedge F , \quad (\text{A.1})$$

where  $f$  denotes the world volume flux. A key point here is that the two-cycles  $\omega_\alpha$  need not be globally homologically non-trivial, in which case there is no propagating field associated with  $C_2^\alpha$  and no Green-Schwarz coupling. As this property requires a global completion to determine, this corresponds to the  $\mathcal{N} = 2$  twisted sector fields being non-normalisable in the non-compact geometry as discussed in this appendix. At the CFT level this corresponds to a logarithmic divergence, which may either vanish or be enhanced to a quadratic divergence in the presence of the global completion.

Multiplicity	Representation
2	$(n_0, \bar{n}_1)$
2	$(n_1, \bar{n}_2)$
2	$(n_2, \bar{n}_3)$
2	$(n_3, \bar{n}_0)$
1	$(n_0, \bar{n}_2)$
1	$(n_2, \bar{n}_0)$
1	$(n_1, \bar{n}_3)$
1	$(n_3, \bar{n}_1)$

**Table 4:** Field content and representations for  $Z_4$  orbifold. The bracket pairs are bi-fundamental representations.

The massless fermionic spectrum of the theory can be calculated using (2.2) which gives the matter content shown in table 4. The non-abelian anomalies of the theory correspond to (A.4). We are also interested in the abelian and mixed anomalies. We consider a general  $U(1)$  combination

$$U(1)_Y \equiv Y_0 U(1)_0 + Y_1 U(1)_1 + Y_2 U(1)_2 + Y_3 U(1)_3 . \quad (\text{A.5})$$

Then, after imposing the tadpoles, the mixed and abelian gauge anomalies are given by

$$\begin{aligned} \mathcal{A}_{SU(n_0)^2-U(1)_Y} &= -\mathcal{A}_{SU(n_2)^2-U(1)_Y} = 2n_1(Y_3 - Y_1) \\ \mathcal{A}_{SU(n_1)^2-U(1)_Y} &= -\mathcal{A}_{SU(n_3)^2-U(1)_Y} = 2n_0(Y_0 - Y_2) \\ \mathcal{A}_{U(1)_Y^3} &= -6n_0n_1 [(Y_1 - Y_3)(Y_0^2 - Y_2^2) - (Y_0 - Y_2)(Y_1^2 - Y_3^2)] . \end{aligned} \quad (\text{A.6})$$

To extract the relevant coupling we turn on the background field

$$q_1 = -q_2 = \frac{1}{\mathcal{N}} (Y_0 \mathbf{1}_{n_0}, Y_1 \mathbf{1}_{n_1}, Y_2 \mathbf{1}_{n_2}, Y_3 \mathbf{1}_{n_3}) , \quad (\text{A.7})$$

where  $\mathcal{N} = \sqrt{2 \sum_{a=0}^3 Y_a^2 n_a}$ .

### The $\mathcal{N} = 1$ sector

The  $B^2$  coefficient for the Annulus amplitude in the UV (C.8) for the  $\mathcal{N} = 1$  sector can be decomposed as

$$\mathcal{A}_{\mathcal{N}=1}^{B^2} = \mathcal{A}_{\mathcal{N}=1}^{q_1^2+q_2^2} + \mathcal{A}_{\mathcal{N}=1}^{q_1 q_2} . \quad (\text{A.8})$$

The amplitude  $\mathcal{A}_{\mathcal{N}=1}^{q_1^2+q_2^2}$  corresponds to the tadpoles exactly as in the case of the non-abelian generators studied in the main sections. The amplitude  $\mathcal{A}_{\mathcal{N}=1}^{q_1 q_2}$  vanished for the non-abelian case because the generators were traceless. However it is non vanishing for the abelian case and in the UV gives the divergence associated with a  $C_2 \wedge F$  coupling in the action with  $C_2$  being an  $\mathcal{N} = 1$  twisted RR mode in this case. In the closed string channel the amplitude reads

$$\mathcal{A}_{\mathcal{N}=1}^{q_1 q_2} \xrightarrow{l' \rightarrow \infty} \left( \frac{B}{4\pi^2} \right)^2 \int_{l'}^{\infty} dl \frac{8}{\mathcal{N}^2} \left[ n_0^2 (Y_0 - Y_2)^2 + n_1^2 (Y_1 - Y_3)^2 \right] . \quad (\text{A.9})$$

We see that this gives precisely the coupling needed to cancel mixed anomalies, and half of the appropriate expressions for the cubic abelian anomalies (A.6). Any  $U(1)_Q$  for which this amplitude is non-vanishing gains a mass. The  $B^4$  amplitude gives the coupling  $C_0 F \wedge F$  and reads

$$\mathcal{A}_{\mathcal{N}=1}^{q_1^2 q_2^2} \xrightarrow{\nu \rightarrow \infty} -B^2 \left( \frac{B}{4\pi^2} \right)^2 \int_{\nu}^{\infty} dl \frac{2}{\mathcal{N}^4} \left[ n_0^2 (Y_0^2 - Y_2^2)^2 + n_1^2 (Y_1^2 - Y_3^2)^2 \right] . \quad (\text{A.10})$$

This takes the form required to match the expression for the cubic anomalies. We have therefore checked that the  $\mathcal{N} = 1$  RR field couples in the correct way to cancel the abelian anomalies.

### The $\mathcal{N} = 2$ sector

The closed string  $\mathcal{N} = 2$  sector is sensitive to the global geometry. Extracting the coupling by studying the UV divergence of the  $B^2$  and  $B^4$  amplitudes is more complicated since it depends on the global completion of the local model. This is reflected in terms of winding modes affecting the amplitude above the winding scale. Indeed the local, in the sense of not including winding modes, amplitudes take the form

$$\mathcal{A}_{\mathcal{N}=2}^{q_1 q_2} \xrightarrow{\nu \rightarrow \infty} \left( \frac{B}{4\pi^2} \right)^2 \int_{\nu}^{\infty} \frac{dl}{l} \frac{8}{\mathcal{N}^2} [n_0 (Y_0 + Y_2) - n_1 (Y_1 + Y_3)]^2 , \quad (\text{A.11})$$

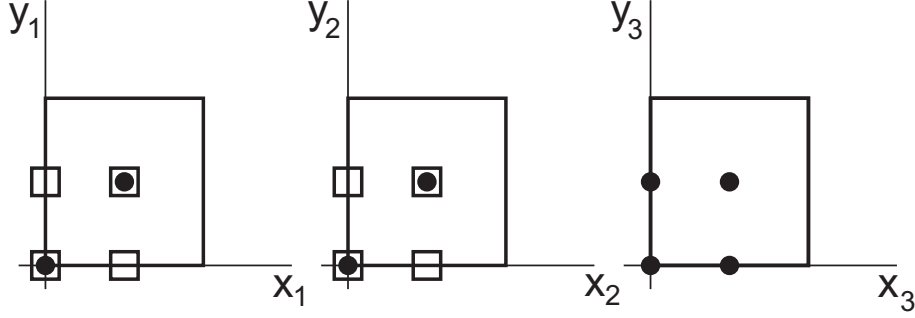
$$\mathcal{A}_{\mathcal{N}=2}^{q_1^2 q_2^2} \xrightarrow{\nu \rightarrow \infty} -B^2 \left( \frac{B}{4\pi^2} \right)^2 \int_{\nu}^{\infty} \frac{dl}{l} \frac{1}{\mathcal{N}^4} [n_0 (Y_0^2 + Y_2^2) - n_1 (Y_1^2 + Y_3^2)]^2 . \quad (\text{A.12})$$

The  $\mathcal{N} = 2$  divergence is logarithmic rather than linear, as was the case for the  $\mathcal{N} = 1$  sector, which reflects the fact that the closed string  $\mathcal{N} = 2$  modes are not propagating physical modes without a global completion. This implies that they cannot participate in the anomaly cancellation since this is a local property. However they can still induce a Green-Schwarz mass for  $U(1)$  fields depending on the global completion. Indeed we see that the dependence of the amplitudes on the  $Y_a$  is not directly related to the field theory anomalies.

We now want to investigate the physics once we compactify the space. As our testbed we will use the  $T^6/\mathbb{Z}_4$  orbifold shown in figure A. We will introduce an additional brane stack to cancel twisted tadpoles. We will not cancel untwisted tadpoles; while clearly this is necessary in a full compactification it does not affect the physics of interest here. As a compact space this orbifold has  $h^{1,1} = 31, h^{2,1} = 7$ . The 31 elements of  $h^{1,1}$  decomposes as 5 untwisted 2-cycles, 16  $\theta^1$  twisted cycles stuck at the 16  $\mathbb{Z}_4$  fixed points, 6  $\theta^2$  twisted cycles stuck at  $\mathbb{Z}_4$  invariant combinations of  $\theta^2$  fixed points, and 4  $\theta^2$  twisted cycles at  $\mathbb{Z}_4$  fixed points and propagating across the third  $T^2$ .

We place a single stack of fractional branes at the origin  $(0, 0, 0)$  (point A) of multiplicity  $(n_0, n_1, n_2, n_3) = (N, M, N, M)$ . As in [7] we also introduce a stack of fractional branes on the  $(0, 0, i/2)$  (point B) of multiplicity  $(n_0, n_1, n_2, n_3) = (M, N, M, N)$ . This cancels the  $\mathcal{N} = 2$  twisted tadpoles. The effect of the compact space and additional stack of branes is





**Figure 2:** The  $T^6/\mathbb{Z}_4$  orbifold. Dark circles correspond to  $\theta$  fixed points and hollow squares correspond to  $\theta^2$  fixed points.

to modify the  $q_1 q_2$  amplitude. We must now include the  $AA$  winding modes which give an extra factor  $\sum_{n,m} e^{-\pi R^2 t(n^2+m^2)}$ .<sup>10</sup> In the UV limit  $l \rightarrow \infty$  this gives

$$\sum_{n,m} e^{-\pi R^2 t(n^2+m^2)} \rightarrow \frac{2l}{R^2} \left(1 + \mathcal{O}(e^{-\frac{2\pi l}{R^2}})\right), \quad (\text{A.13})$$

The amplitude therefore becomes

$$\mathcal{A}_{\mathcal{N}=2}^{q_1 q_2} \xrightarrow{l' \rightarrow \infty} \left(\frac{B}{4\pi^2}\right)^2 \int_{l'}^{\infty} \frac{dl}{l} \frac{2l}{R^2} \frac{8}{\mathcal{N}^2} [n_0(Y_0 + Y_2) - n_1(Y_1 + Y_3)]^2. \quad (\text{A.14})$$

This is now linearly divergent and corresponds to a physical coupling

$$\frac{m_s}{R} (n_0(Y_0 + Y_2) - n_1(Y_1 + Y_3)) C_2 \wedge F, \quad (\text{A.15})$$

which induces a mass for the field.

We can now determine the masses of the non-anomalous  $U(1)$ s in the model. There are two orthogonal non-anomalous  $U(1)$ s to consider

$$U(1)_{diag} = \frac{1}{n_0}(U(1)_0 + U(1)_2) + \frac{1}{n_1}(U(1)_1 + U(1)_3), \quad (\text{A.16})$$

$$U(1)_{tw} = \frac{1}{n_1}(U(1)_0 + U(1)_2) - \frac{1}{n_0}(U(1)_1 + U(1)_3). \quad (\text{A.17})$$

From (A.15) we see that  $U(1)_{diag}$  remains massless while  $U(1)_{tw}$  gains a mass.

Actually within a global context  $C_2$  can couple to both the  $A$  and  $B$  stacks of branes. We therefore ought to consider a general  $U(1)$  which combines both stacks of branes,  $U(1) = \sum_a Y_a U(1)_a^A + Z_a U(1)_a^B$ . As neither  $U(1)_{diag}^A$  nor  $U(1)_{diag}^B$  have any coupling to the twisted sectors, the same is true of linear combinations of these, which therefore remain massless in the compact model. The interesting case, which we focus on, is the combination

<sup>10</sup>There is also potentially an  $AB$  winding mode stack with a factor  $\sum_{n,m} e^{-\pi R^2 t((n+1/2)^2+m^2)}$ , which does not contribute to the  $q_1 q_2$  sector.

$U(1)_{tw}^A \pm U(1)_{tw}^B$ . We can in fact verify that for

$$U(1)_X = \left( \frac{1}{n_1}(U(1)_0 + U(1)_2) - \frac{1}{n_0}(U(1)_1 + U(1)_3) \right) - \left( \frac{1}{n_0}(U(1)_0 + U(1)_2) - \frac{1}{n_1}(U(1)_1 + U(1)_3) \right), \quad (\text{A.18})$$

then  $\mathcal{A}_{\mathcal{N}=2}^{q_1 q_2} \rightarrow 0$ . This implies that the amplitude has no quadratic divergence and  $U(1)_X$  has no  $C_2 \wedge F$  coupling to give it a mass. In contrast  $U(1)_{tw}^A + U(1)_{tw}^B$  has a nonvanishing  $\mathcal{A}_{\mathcal{N}=2}^{q_1 q_2}$  and becomes massive with a mass given by  $\sim m_s/R$  consistent with [26, 27].

In total there are then three massless orthogonal  $U(1)$ s present:

$$\begin{aligned} U(1)_{diag}^A &= \frac{1}{n_0}(U(1)_0^A + U(1)_2^A) + \frac{1}{n_1}(U(1)_1^A + U(1)_3^A), \\ U(1)_{diag}^B &= \frac{1}{n_1}(U(1)_0^B + U(1)_2^B) + \frac{1}{n_0}(U(1)_1^B + U(1)_3^B), \\ U(1)_X &= \left( \frac{1}{n_1}(U(1)_0^A + U(1)_2^A) - \frac{1}{n_0}(U(1)_1^A + U(1)_3^A) \right) \\ &\quad - \left( \frac{1}{n_0}(U(1)_0^B + U(1)_2^B) - \frac{1}{n_1}(U(1)_1^B + U(1)_3^B) \right). \end{aligned} \quad (\text{A.19})$$

The third  $U(1)$  mixes the ‘visible’ and ‘hidden’ sector. The details of this  $U(1)$ , and the fields that are charged under it, depend on the precise nature of the hidden sector.

This shows explicitly at the CFT level how the masses of non-anomalous  $U(1)$ s are determined depending on the global geometry. Analysed locally, we obtain a logarithmic divergence. If the cycle is globally trivial, then as we extend to a fully global model the divergence vanishes and the Green-Schwarz coupling is absent. If the cycle is non-trivial, then the logarithmic divergence becomes a quadratic divergence suppressed by the bulk radius. This quadratic divergence signals the presence of the Green-Schwarz term and the  $U(1)$  mass.

The same techniques using the background field formalism should be applicable for the related problem of studying kinetic mixing among separate  $U(1)$ s which may be interesting to carry out. This would be complementary to the vertex operator techniques used in [28].

## B. Tadpoles

In this appendix we calculate the tadpole divergences for local orientifolds. We are interested in  $\mathcal{N} = 1$  fully twisted tadpoles that receive contributions from the annulus, Mobius strip, and Klein bottle one-loop amplitudes. The annulus amplitude is given by (we work in units with  $2\alpha' = 1$ )

$$\mathcal{A} = - \int_0^\infty \frac{dt}{2t} \text{STr} \left[ \frac{1}{N} \sum_{k=0}^N \theta^k \left( \frac{1 + (-1)^F}{2} \right) q^{(p_\mu p^\mu + M^2)/2} \right]. \quad (\text{B.1})$$

Here  $t$  parameterises the annulus width with  $t \rightarrow 0$  corresponding to the UV and  $t \rightarrow \infty$  the IR. The sum over  $\theta^k$  imposes the orbifold projection and  $\left( \frac{1 + (-1)^F}{2} \right)$  the GSO projection.

$q = e^{2\pi i \tau} = e^{-\pi t}$  where  $\tau$  is the torus parameter for the annulus  $\tau = it/2$ .  $p_\mu$  and  $M$  are the momentum and mass of the string states. The STr stands for tracing over the bosons and fermions as  $\sum_{\text{NS}} - \sum_{\text{RR}}$ . Performing this trace we can write

$$\mathcal{A} = \frac{1}{N} \sum_{k=0}^N \mathcal{A}^{(k)}, \quad (\text{B.2})$$

with

$$\mathcal{A}_{\mathcal{N}=1}^{(k)} = - \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)^2} \text{Tr} [\gamma_k \otimes \gamma_k^{-1}] \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^3 \left( -2 \sin \pi \theta_i^k \right) \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + \theta_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + \theta_i^k \end{smallmatrix} \right]}. \quad (\text{B.3})$$

Here the  $\eta$  and  $\theta$  functions are functions of  $q$  and are explicitly given in appendix E. We have  $\eta_{\alpha\beta} = (-1)^{2(\alpha+\beta-2\alpha\beta)}$ . The trace is coming from the trace over the CP indices. The terms with  $\alpha = 0$  and  $\alpha = 1/2$  are due to RR and NS states respectively.

Similarly we have for the Mobius strip and Klein bottle

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} = \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)^2} \text{Tr} [\gamma_{\Omega'_k} \gamma_{\Omega_k'^{-T}}] \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^3 \left( -2 \sin (\pi R_i^k) \right) \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + R_i^k \end{smallmatrix} \right]}, \quad (\text{B.4})$$

$$\mathcal{K}_0^{(k)} = 4 \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)^2} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^3 \left( \frac{-2 \sin (2\pi R_i^k)}{4 \cos^2 (\pi R_i^k)} \right) \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + 2R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + 2R_i^k \end{smallmatrix} \right]}, \quad (\text{B.5})$$

$$\mathcal{K}_2^{(k)} = 4 \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)^2} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^3 \left( \frac{-2 \sin (2\pi R_i^k)}{4 \cos^2 (\pi R_i^k)} \right)^{\delta_i} \frac{\vartheta \left[ \begin{smallmatrix} \alpha + \theta_i^{N/2} \\ \beta + 2R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 + \theta_i^{N/2} \\ 1/2 + 2R_i^k \end{smallmatrix} \right]}. \quad (\text{B.6})$$

Here

$$R_i^k = \theta_i^k + R_i. \quad (\text{B.7})$$

The last amplitude  $\mathcal{K}_2^{(k)}$  only occurs for even orientifolds and in that case

$$\delta_i = \begin{cases} 0 & \text{if } \theta_i^{N/2} \bmod 1 = 1/2 \\ 1 & \text{otherwise} \end{cases}. \quad (\text{B.8})$$

It corresponds to the contribution for the  $Z_2$  twisted closed string sector with all other contributions, apart from the untwisted sector  $\mathcal{K}_0^{(k)}$ , vanishing by symmetry. The Mobius strip amplitude is a function of  $-q$  and the Klein bottle of  $q^2$  corresponding to torus

parameters of  $\tau = 1/2 + it/2$  and  $\tau = it$  respectively. The extra factors of  $\cos^2$  in the denominators are due to zero mode integration over the internal space [29].

To extract the UV divergence we can modular transform, using formulae in appendix E, to the (tree level) closed string channel through the transformations  $t = 1/l$ ,  $t = 1/4l$ ,  $t = 1/2l$  for the annulus, Mobius strip and Klein bottle respectively.  $l$  is now the cylinder length and the UV limit is given by  $l \rightarrow \infty$ . Performing the transformation we find<sup>11</sup>

$$\mathcal{A}_{\mathcal{N}=1}^{(k)} = \frac{i}{4} \int \frac{dl}{(2\pi^2)^2} \text{Tr} [\gamma_k \otimes \gamma_k^{-1}] \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}}{\eta^3} \prod_{i=1}^3 (-2 \sin \pi \theta_i) \frac{\vartheta \begin{bmatrix} \beta + \theta_i \\ -\alpha \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + \theta_i \\ -1/2 \end{bmatrix}} \quad (\text{B.9})$$

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} = -2i \int \frac{dl}{(2\pi^2)^2} \text{Tr} [\gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T}] \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}}{\eta^3} \prod_{i=1}^3 \left( -2 \sin \left( \pi R_i^k \right) \right) \frac{\vartheta \begin{bmatrix} \alpha + 2R_i^k \\ \beta + R_i^k \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + 2R_i^k \\ 1/2 + R_i^k \end{bmatrix}}. \quad (\text{B.10})$$

$$\mathcal{K}_0^{(k)} = -4i \int \frac{dl}{(2\pi^2)^2} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}}{\eta^3} \frac{\prod_{i=1}^3 (-2 \sin 2\pi R_i^k)}{\prod_{i=1}^3 4 \cos^2 (\pi R_i^k)} \frac{\vartheta \begin{bmatrix} \beta + 2R_i^k \\ -\alpha \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + 2R_i^k \\ -1/2 \end{bmatrix}}. \quad (\text{B.11})$$

$$\mathcal{K}_2^{(k)} = -4i \int \frac{dl}{(2\pi^2)^2} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \frac{\vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}}{\eta^3} \frac{\prod_{i=1}^3 (-2 \sin 2\pi R_i^k)^{\delta_i}}{\prod_{i=1}^3 (4 \cos^2 (\pi R_i^k))^{\delta_i}} \frac{\vartheta \begin{bmatrix} \beta + 2R_i^k \\ -\alpha - \theta_i^{N/2} \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + 2R_i^k \\ -1/2 - \theta_i^{N/2} \end{bmatrix}}. \quad (\text{B.12})$$

Here the annulus and Klein bottle amplitudes are functions of  $\tilde{q} = e^{-4\pi l}$  while the Mobius strip is a function of  $-\tilde{q}$ .

The RR tadpoles are given in tree channel by  $\alpha = 0$ ,  $\beta = \frac{1}{2}$  for the annulus and Klein bottle and  $\alpha = \frac{1}{2}$ ,  $\beta = 0$  for the Mobius strip. In the UV limit  $l \rightarrow \infty$  the amplitudes read

$$\mathcal{A}_{\mathcal{N}=1}^{(k)} \xrightarrow{l' \rightarrow \infty} - \int_{l'}^{\infty} \frac{dl}{4\pi^2} \frac{1}{4} \text{Tr} [\gamma_k] \text{Tr} [\gamma_k^{-1}] \prod_{i=1}^3 \left| 2 \sin \pi \theta_i^k \right|, \quad (\text{B.13})$$

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow{l' \rightarrow \infty} \int_{l'}^{\infty} \frac{dl}{4\pi^2} \left[ 2 \text{Tr} [\gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T}] \prod_{i=1}^3 s_i \left( 2 \sin \left( \pi R_i^k \right) \right) \right], \quad (\text{B.14})$$

$$\mathcal{K}_0^{(k)} + \mathcal{K}_2^{(k)} \xrightarrow{l' \rightarrow \infty} - \int_{l'}^{\infty} \frac{dl}{4\pi^2} 4 \left[ \prod_{i=1}^3 \left| \frac{\sin \pi R_i^k}{\cos \pi R_i^k} \right| + (-1)^M \prod_{i=1}^3 (-1)^{\delta_i} \left| \frac{\sin \pi R_i^k}{\cos \pi R_i^k} \right|^{\delta_i} \right] \quad (\text{B.15})$$

where  $s_i = \text{sgn} [\sin(2\pi R_i^k)]$ . The  $(-1)^M$  factor is discussed in footnote 4.

<sup>11</sup>Since the Mobius amplitude is a function of  $-q$  the transformation needs to be done through a series of transformations of the torus parameter  $1/2 + it/2 = \tau \rightarrow -\frac{1}{\tau} \rightarrow -\frac{1}{\tau} + 2 \rightarrow (\frac{1}{\tau} - 2)^{-1} = 2il - 1/2$  [17].

### C. Magnetised amplitudes

In this appendix we calculate the Annulus and Mobius strip magnetised amplitudes. We begin with the annulus amplitudes. The Annulus amplitudes in the background of a magnetic field is given by [7]

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=1}^{(k)} = & - \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left( \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right) \\ & \times \prod_{i=1}^3 \frac{(-2 \sin \pi \theta_i^k) \vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + \theta_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + \theta_i^k \end{smallmatrix} \right]}. \end{aligned} \quad (\text{C.1})$$

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=2}^{(k)} = & - \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} (-1)^{2\alpha} \text{Tr} \left( \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right) \\ & \times \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^2 \frac{(-2 \sin \pi \theta_i^k) \vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + \theta_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + \theta_i^k \end{smallmatrix} \right]}. \end{aligned} \quad (\text{C.2})$$

$$\mathcal{A}_{\mathcal{N}=4}^{(k)} = - \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left[ \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right] \left( \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \right)^3. \quad (\text{C.3})$$

The IR behaviour  $t \rightarrow \infty$  of the amplitudes is used in the main part of the paper to calculate the  $\beta$  functions. These can be extracted to order  $B^2$  directly from the above amplitudes using the methods given in [7, 17]. In this appendix we calculate the UV

behaviour. To do this we first transform to the closed string channel which gives

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=1}^{(k)} &= \int \frac{dl}{4\pi^2} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left( \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \beta \\ -\alpha \end{smallmatrix} \right] (\epsilon \mid 4l)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right] (\epsilon \mid 4l)} \right) \\ &\quad \times \prod_{i=1}^3 \left( -2 \sin \pi \theta_i^k \right) \frac{\vartheta \left[ \begin{smallmatrix} \beta + \theta_i^k \\ -\alpha \end{smallmatrix} \right] (4l)}{\vartheta \left[ \begin{smallmatrix} 1/2 + \theta_i^k \\ -1/2 \end{smallmatrix} \right] (4l)}, \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=2}^{(k)} &= i \int \frac{dl}{4\pi^2 l} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} (-1)^{2\alpha} \text{Tr} \left( \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \beta \\ -\alpha \end{smallmatrix} \right] (\epsilon \mid 4l)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right] (\epsilon \mid 4l)} \right) \\ &\quad \times \frac{\vartheta \left[ \begin{smallmatrix} \beta \\ -\alpha \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^2 \left( -2 \sin \pi \theta_i^k \right) \frac{\vartheta \left[ \begin{smallmatrix} \beta + \theta_i^k \\ -\alpha \end{smallmatrix} \right] (4l)}{\vartheta \left[ \begin{smallmatrix} 1/2 + \theta_i^k \\ -1/2 \end{smallmatrix} \right] (4l)} e^{2\pi i \theta_i^k (\alpha - 1/2)}. \end{aligned} \quad (\text{C.5})$$

$$\mathcal{A}_{\mathcal{N}=4}^{(k)} = -i \int \frac{dl}{4\pi^2} \frac{1}{(2l)^3} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left( \frac{i(\beta_1 + \beta_2)}{2\pi^2} \frac{\vartheta \left[ \begin{smallmatrix} \beta \\ -\alpha \end{smallmatrix} \right] (\epsilon \mid 4l)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] (\epsilon \mid 4l)} \right) \left( \frac{\vartheta \left[ \begin{smallmatrix} \beta \\ -\alpha \end{smallmatrix} \right]}{\eta^3} \right)^3 \quad (\text{C.6})$$

Now we can take the UV limit  $l \rightarrow \infty$  which gives

$$\begin{aligned} \mathcal{A}_{\mathcal{N}=1}^{(k)} &\xrightarrow{l \rightarrow \infty} - \int_{l'}^{\infty} \frac{1}{2} \frac{dl}{4\pi^2} \text{Tr} \left[ \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \frac{i(\beta_1 + \beta_2)}{2\pi^2} \left( i (\cot (\pi\epsilon) - \text{cosec } (\pi\epsilon)) \prod_{i=1}^3 \left| 2 \sin \pi \theta_i^k \right| \right. \right. \\ &\quad \left. \left. + \prod_{i=1}^3 \left( -2 \sin \pi \theta_i^k \right) \right) \right]. \\ \mathcal{A}_{\mathcal{N}=2}^{(k)} &\xrightarrow{l \rightarrow \infty} - \int_{l'}^{\infty} \frac{dl}{4\pi^2 l} \text{Tr} \left( \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \frac{i(\beta_1 + \beta_2)}{2\pi^2} i (\cot (\pi\epsilon) - \text{cosec } (\pi\epsilon)) \prod_{i=1}^2 \left| 2 \sin \pi \theta_i^k \right| \right). \\ \mathcal{A}_{\mathcal{N}=4}^{(k)} &\xrightarrow{l \rightarrow \infty} - \int_{l'}^{\infty} \frac{dl}{4\pi^2} \frac{1}{(2l)^3} \text{Tr} \left[ \frac{i(\beta_1 + \beta_2)}{2\pi^2} \left( -4i \cot (\pi\epsilon) + \frac{i (\cos (2\pi\epsilon) + 3)}{\sin (\pi\epsilon)} \right) \right], \end{aligned} \quad (\text{C.7})$$

Note that these can be further simplified by using  $(\beta_1 + \beta_2) \cot (\pi\epsilon) = 1 - \beta_1 \beta_2$ . We can expand these expressions in powers of the magnetic field  $B$  which gives up to order  $B^4$  the

expressions

$$\begin{aligned}
\mathcal{A}_{\mathcal{N}=1}^{(k)} &\xrightarrow{l' \rightarrow \infty} -\frac{1}{8} \int_{l'}^{\infty} \frac{dl}{(4\pi^2)^2} \text{Tr} \left[ \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \left( 4(\beta_1 + \beta_2) \prod_{i=1}^3 \left( -2 \sin \pi \theta_i^k \right) \right. \right. \\
&\quad \left. \left. + \left( 4(\beta_1 + \beta_2)^2 - (\beta_1^2 - \beta_2^2)^2 \right) \prod_{i=1}^3 \left| 2 \sin \pi \theta_i^k \right| \right) \right] , \\
\mathcal{A}_{\mathcal{N}=2}^{(k)} &\xrightarrow{l' \rightarrow \infty} -\frac{1}{2} \int_{l'}^{\infty} \frac{dl}{(4\pi^2)^2} \frac{1}{(2l)} \text{Tr} \left[ \gamma_{\theta^k} \otimes \gamma_{\theta^k}^{-1} \left( 4(\beta_1 + \beta_2)^2 - (\beta_1^2 - \beta_2^2)^2 \right) \right] \prod_{i=1}^2 \left| 2 \sin \pi \theta_i^k \right| , \\
\mathcal{A}_{\mathcal{N}=4}^{(k)} &\xrightarrow{l' \rightarrow \infty} \int_{l'}^{\infty} \frac{dl}{(4\pi^2)^2} \frac{1}{(2l)^3} \text{Tr} \left[ (\beta_1 + \beta_2)^4 \right] . \tag{C.8}
\end{aligned}$$

Now we repeat the same calculations for the  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  magnetised Mobius strip amplitudes. In the open string loop channel we have

$$\begin{aligned}
\mathcal{M}_{\mathcal{N}=1}^{(k)} &= 2 \int_0^{\infty} \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left[ \frac{i}{2\pi^2} \beta \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right] \prod_{i=1}^3 \left( -2 \sin \left( \pi R_i^k \right) \right) \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + R_i^k \end{smallmatrix} \right]} . \\
\mathcal{M}_{\mathcal{N}=2}^{(k)} &= 2 \int \frac{dt}{2t} \frac{1}{(2\pi^2 t)} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} (-1)^{2\alpha} \text{Tr} \left( \frac{i}{2\pi^2} \beta \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{i\epsilon t}{2} \right)} \right) \\
&\quad \times \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right]}{\eta^3} \prod_{i=1}^2 \frac{(-2 \sin \pi R_i^k) \vartheta \left[ \begin{smallmatrix} \alpha \\ \beta + R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 + R_i^k \end{smallmatrix} \right]} . \tag{C.9}
\end{aligned}$$

Again the IR behaviour can be extracted as for the Annulus. In the main section we are interested in the UV behaviour for the  $\mathcal{N} = 1$  sectors. Transforming to the tree channel we find

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} = -8 \int \frac{dl}{4\pi^2} \sum_{\alpha, \beta=0, 1/2} \frac{\eta_{\alpha\beta}}{2} \text{Tr} \left[ \frac{i}{2\pi^2} \beta \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \frac{\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left( \frac{\epsilon}{2} \right)}{\vartheta \left[ \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \left( \frac{\epsilon}{2} \right)} \right] \prod_{i=1}^3 \left( -2 \sin \left( \pi R_i^k \right) \right) \frac{\vartheta \left[ \begin{smallmatrix} \alpha + 2R_i^k \\ \beta + R_i^k \end{smallmatrix} \right]}{\vartheta \left[ \begin{smallmatrix} 1/2 + 2R_i^k \\ 1/2 + R_i^k \end{smallmatrix} \right]} . \tag{C.10}$$

Taking the UV limit we get

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow{l' \rightarrow \infty} 4 \int_{l'}^{\infty} \frac{dl}{4\pi^2} \text{Tr} \left[ \frac{i}{2\pi^2} \beta \gamma_{\Omega'_k} \gamma_{\Omega'_k}^{-T} \left( 1 - i \left( \cot \left( \pi \epsilon / 2 \right) - \frac{1}{\sin \left( \pi \epsilon / 2 \right)} \right) \prod_{i=1}^3 s_i \right) \right] \prod_{i=1}^3 \left( -2 \sin \left( \pi R_i^k \right) \right) . \tag{C.11}$$

We can expand this as

$$\mathcal{M}_{\mathcal{N}=1}^{(k)} \xrightarrow{l' \rightarrow \infty} -4 \int_{l'}^{\infty} \frac{dl}{(4\pi^2)^2} \text{Tr} \left[ \left( \beta^2 + \frac{1}{4} \beta^4 \right) \gamma_{\Omega'_k} \gamma_{\Omega'^{-T}_k} \right] \prod_{i=1}^3 s_i \left( -2 \sin \left( \pi R_i^k \right) \right) . \quad (\text{C.12})$$

## D. Chiral Vs Linear multiplets at 1-loop

In this appendix we discuss the dualisation of linear multiplets to chiral multiplets in supergravity. The analysis follows that presented in [18, 19] but applied to the local IIB models studied in this paper. For more details regarding the supergravity constructions we refer to [19].

The linear multiplet  $L$  is defined by the constraint

$$(D^2 - 8\bar{R}) L = (\bar{D}^2 - 8R) L = 0 , \quad (\text{D.1})$$

where  $D$  is the superspace covariant derivative and  $R$  is the chiral superfield containing the curvature scalar. The bosonic components are a real scalar, which we denote again by  $L$ , and a real two-form  $\tilde{C}_2$ . We can couple  $L$  to a Yang-Mills (YM) gauge field  $\mathcal{A}$  with field strength  $\mathcal{F}$  through a Green-Schwarz coupling  $\tilde{C}_2 \wedge \mathcal{F}$ . This implies that the field strength of  $\tilde{C}_2$  is modified in order to be gauge invariant under the YM gauge transformation

$$\tilde{F} = d\tilde{C}_2 + k\Omega , \quad (\text{D.2})$$

where  $k$  is a constant and  $\Omega$  is the Chern-Simons form  $\Omega = \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$ . This modifies the Linear multiplet constraints (D.1) to

$$(D^2 - 8\bar{R} - 2k\text{Tr}\mathcal{W}^2) L = (\bar{D}^2 - 8R - 2k\text{Tr}\bar{\mathcal{W}}^2) L = 0 , \quad (\text{D.3})$$

with  $\text{Tr}\mathcal{W}^2 = \frac{1}{2} (\bar{D}^2 - 8R) \Omega$  the usual YM field strength. The supergravity action for single linear multiplet  $L$  and chiral multiplets (collectively denoted as)  $T$  is given by

$$S = -3 \int EF(T, \bar{T}, L) + \left\{ \frac{1}{2} \int \frac{E}{R} e^{K/2} W(T) + \text{hc} \right\} + \dots , \quad (\text{D.4})$$

where  $E$  is the super-vielbein,  $W$  the superpotential, and  $K(T, \bar{T}, L)$  is the Kahler potential. The function  $F(T, \bar{T}, L)$  is called the *subsidiary function* and is related to the Kahler potential through

$$F - LF_L = 1 - \frac{1}{3} LK_L , \quad (\text{D.5})$$

where subscripts denote derivatives. Equation (D.5) comes from the fact that in the linear multiplet formalism integrating over the superdeterminant gives a non-canonically normalised Einstein term and (D.5) is the correct normalisation constraint. The constraint (D.5) fixes

$$F(T, \bar{T}, L) = 1 + L\Delta(T, \bar{T}) + \frac{L}{3} \int \frac{d\lambda}{\lambda} K_\lambda(T, \bar{T}, \lambda) , \quad (\text{D.6})$$



with  $\lambda$  a dummy variable for  $L$ .  $\Delta(T, \bar{T})$  is call the *Linear potential* and forms part of the gauge coupling of  $L$  to  $\mathcal{A}$ . which is given by [19]

$$3k \left( \frac{F(T, \bar{T}, L) - 1}{L} \right) \text{Tr} \mathcal{F}^2. \quad (\text{D.7})$$

In our case  $\Delta(T, \bar{T})$  will encode the 1-loop correction to the gauge coupling associated to the field  $L$ .

We wish to dualise the linear multiplet  $L$  to a chiral multiplet  $M$  (which has a propagating bosonic component of single complex scalar field  $M$ ). The  $\text{Re}M$  is dual to the scalar  $L$  and  $\text{Im}M$  is dual to  $\tilde{C}_2$ . To do this we introduce the coupling to the action

$$S = -3 \int E [F(T, \bar{T}, L) + (L - k\Omega)(M + \bar{M})] + \dots. \quad (\text{D.8})$$

Then we find the equation of motion for  $L$  which read<sup>12</sup>

$$(M + \bar{M}) \left( 1 - \frac{1}{3} L K_L \right) = \frac{1}{3} F K_L - F_L. \quad (\text{D.9})$$

This equation can be used in principle to solve for  $L(M + \bar{M}, T, \bar{T})$  and write the action using only chiral superfields. Now using (D.5) and (D.9) we find

$$F(T, \bar{T}, L) + L(M + \bar{M}) = 1. \quad (\text{D.10})$$

Substituting this into the action (D.8) gives

$$\begin{aligned} S &= -3 \int E [1 - k\Omega(M + \bar{M})] + \dots \\ &= -3 \int E - \left\{ \frac{3}{8} k \int \frac{E}{R} M (\bar{\mathcal{D}}^2 - 8R) \Omega + \text{hc} \right\} + \dots \\ &= -3 \int E - \left\{ \frac{3}{4} k \int \frac{E}{R} M \text{Tr} \mathcal{W}^2 + \text{hc} \right\} + \dots. \end{aligned} \quad (\text{D.11})$$

Here in the first step the derivative terms vanish upon integration by parts and we used the expression below (D.3) in the second step. This implies that in the chiral multiplet formalism the familiar holomorphic gauge kinetic function  $f(M)$  takes the form

$$f = -6kM. \quad (\text{D.12})$$

Equivalently this can be derived by simply substituting (D.10) into (D.7) and using holomorphy. This should be compared with what is denoted the tree-level gauge kinetic function in the main text (1.3). We see that they match and so what remains is to determine the precise relationship between  $\text{Re}M$  and  $L$ .

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<sup>12</sup>Note that  $\delta_L E = -\frac{1}{3} E K_L \delta L$  and  $\delta_L \Omega = \frac{1}{3} \Omega K_L \delta L$ .

## D.1 Application to local IIB models

In this subsection we apply the results derived in the previous section to the local IIB models studied in the main text. Of course the analysis of the previous section is vastly over simplified since there are many fields in a concrete construction but the key properties can be deduced considering only one field which is what we do in this subsection.

We begin by specifying the Kahler potential which in local models takes the form

$$K = -2\ln\mathcal{V}(T, \bar{T}) + \mathcal{V}(T, \bar{T}) L^2 + K_0(U, S) , \quad (\text{D.13})$$

where  $U$  and  $S$  are the complex-structure and dilaton fields respectively and  $\mathcal{V}(T, \bar{T})$  denotes the CY volume as a function of the four-cycle volumes  $T$ .<sup>13</sup> This Kahler potential, using (D.6), gives a subsidiary function of

$$F = 1 + L\Delta + \frac{2\mathcal{V}}{3}L^2 . \quad (\text{D.14})$$

Then using (D.13) and (D.10) gives

$$\text{Re}M = -\frac{1}{2} \left( \frac{2\mathcal{V}}{3}L + \Delta \right) . \quad (\text{D.15})$$

This should be compared with (4.4) and we see that they match up to a field rescaling  $m = \frac{2\mathcal{V}}{3}L$ . Using (D.12) we can also match the field coupling  $-6k = s$ .

Finally we need to determine the nature of  $\Delta$ . From (D.14) we see that at the orbifold limit  $L = 0$  the gauge coupling (D.7) takes the form

$$\frac{1}{g^2} = 3k\Delta . \quad (\text{D.16})$$

This shows that the field redefinition is given by the 1-loop correction to the gauge coupling as required in the main text.

## E. Some conventions and formulae

We here collate definitions and identities of the various Jacobi- $\vartheta$  functions. We write  $q = e^{-\pi t}$  throughout these formulae. The eta function is defined by

$$\eta(t) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (\text{E.1})$$

The Jacobi  $\vartheta$ -function with general characterstic is defined as

$$\vartheta \left[ \begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z|t) = \sum_{n \in \mathbb{Z}} q^{(n+\alpha)^2/2} e^{2\pi i(z+\beta)(n+\alpha)}. \quad (\text{E.2})$$

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<sup>13</sup>The factor of  $\mathcal{V}$  in front of the  $L^2$  term is justified from the fact that after dualisation to the chiral multiplet this Kahler potential term takes the form  $K(M, \bar{M}) = \frac{9(M+\bar{M})^2}{4\mathcal{V}}$  which is the form advocated in [27] for the blow-up modulus.

Here  $z = 0$  unless specified. The  $\vartheta$  functions are manifestly invariant under  $\alpha \rightarrow \alpha + \mathbb{Z}$ . A useful expansion valid for  $\alpha \in (-\frac{1}{2}, \frac{1}{2}]$  is

$$\frac{\vartheta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]}{\eta}(t) = e^{2\pi i \alpha \beta} q^{\frac{\alpha^2}{2} - \frac{1}{24}} \prod_{n=1}^{\infty} \left(1 + e^{2\pi i \beta} q^{n - \frac{1}{2} + \alpha}\right) \left(1 + e^{-2\pi i \beta} q^{n - \frac{1}{2} - \alpha}\right). \quad (\text{E.3})$$

For the four special  $\vartheta$ -functions, we have

$$\vartheta_1(z|t) \equiv \vartheta\left[\begin{smallmatrix}\frac{1}{2} \\ \frac{1}{2}\end{smallmatrix}\right](z|t) = 2q^{1/8} \sin \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i z} q^n)(1 - e^{-2\pi i z} q^n). \quad (\text{E.4})$$

$$\vartheta_2(z|t) \equiv \vartheta\left[\begin{smallmatrix}\frac{1}{2} \\ 0\end{smallmatrix}\right](z|t) = 2q^{1/8} \cos \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{2\pi i z} q^n)(1 + e^{-2\pi i z} q^n). \quad (\text{E.5})$$

$$\vartheta_3(z|t) \equiv \vartheta\left[\begin{smallmatrix}0 \\ 0\end{smallmatrix}\right](z|t) = \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{2\pi i z} q^{n - \frac{1}{2}})(1 + e^{-2\pi i z} q^{n - \frac{1}{2}}). \quad (\text{E.6})$$

$$\vartheta_4(z|t) \equiv \vartheta\left[\begin{smallmatrix}0 \\ \frac{1}{2}\end{smallmatrix}\right](z|t) = \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i z} q^{n - \frac{1}{2}})(1 - e^{-2\pi i z} q^{n - \frac{1}{2}}). \quad (\text{E.7})$$

The functions transform as (from [17])

$$\vartheta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right](z|\tau) = e^{-2\pi i \alpha \beta - \frac{i\pi z^2}{\tau}} \sqrt{\frac{i}{\tau}} \vartheta\left[\begin{smallmatrix}-\beta \\ \alpha\end{smallmatrix}\right]\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right), \quad (\text{E.8})$$

$$\vartheta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right](z|\tau) = e^{\pi i \alpha(\alpha - 1)} \vartheta\left[\begin{smallmatrix}\alpha \\ \beta - \alpha + 1/2\end{smallmatrix}\right](z|\tau + 1). \quad (\text{E.9})$$

## References

- [1] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, “Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes,” *Phys. Rept.* **445** (2007) 1 [arXiv:hep-th/0610327].
- [2] G. Aldazabal, L. E. Ibanez, F. Quevedo and A. M. Uranga, “D-branes at singularities: A bottom-up approach to the string embedding of the standard model,” *JHEP* **0008**, 002 (2000) [arXiv:hep-th/0005067].
- [3] V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, “Systematics of Moduli Stabilisation in Calabi-Yau Flux Compactifications,” *JHEP* **0503**, 007 (2005) [arXiv:hep-th/0502058].
- [4] J. P. Conlon, F. Quevedo and K. Suruliz, “Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking,” *JHEP* **0508**, 007 (2005) [arXiv:hep-th/0505076].
- [5] V. S. Kaplunovsky and J. Louis, “Model independent analysis of soft terms in effective supergravity and in string theory,” *Phys. Lett. B* **306**, 269 (1993) [arXiv:hep-th/9303040].
- [6] V. Kaplunovsky and J. Louis, “Field dependent gauge couplings in locally supersymmetric effective quantum field theories,” *Nucl. Phys. B* **422**, 57 (1994) [arXiv:hep-th/9402005].

- [7] J. P. Conlon, “Gauge Threshold Corrections for Local String Models,” arXiv:0901.4350 [hep-th].
- [8] J. P. Conlon, E. Palti, “Gauge Threshold Corrections for Local GUTs,” to appear.
- [9] C. Beasley, J. J. Heckman and C. Vafa, “GUTs and Exceptional Branes in F-theory - I,” JHEP **0901** (2009) 058 [arXiv:0802.3391 [hep-th]].
- [10] R. Donagi and M. Wijnholt, “Model Building with F-Theory,” arXiv:0802.2969 [hep-th].
- [11] A. Abouelsaood, C. G. . Callan, C. R. Nappi and S. A. Yost, “Open Strings In Background Gauge Fields,” Nucl. Phys. B **280** (1987) 599.
- [12] C. Bachas and M. Porrati, “Pair creation of open strings in an electric field,” Phys. Lett. B **296**, 77 (1992) [arXiv:hep-th/9209032].
- [13] C. Bachas and C. Fabre, “Threshold Effects in Open-String Theory,” Nucl. Phys. B **476**, 418 (1996) [arXiv:hep-th/9605028].
- [14] D. Lust and S. Stieberger, “Gauge threshold corrections in intersecting brane world models,” Fortsch. Phys. **55**, 427 (2007) [arXiv:hep-th/0302221].
- [15] R. Blumenhagen, “Gauge Coupling Unification In F-Theory Grand Unified Theories,” Phys. Rev. Lett. **102**, 071601 (2009) [arXiv:0812.0248 [hep-th]].
- [16] R. Donagi and M. Wijnholt, “Breaking GUT Groups in F-Theory,” arXiv:0808.2223 [hep-th].
- [17] I. Antoniadis, C. Bachas and E. Dudas, “Gauge couplings in four-dimensional type I string orbifolds,” Nucl. Phys. B **560** (1999) 93 [arXiv:hep-th/9906039].
- [18] J. P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, “On loop corrections to string effective field theories: Field dependent gauge couplings and sigma model anomalies,” Nucl. Phys. B **372** (1992) 145.
- [19] P. Binetruiy, G. Girardi and R. Grimm, “Supergravity couplings: a geometric formulation,” Phys. Rept. **343** (2001) 255 [arXiv:hep-th/0005225].
- [20] L. E. Ibanez, R. Rabadan and A. M. Uranga, “Anomalous  $U(1)$ ’s in type I and type IIB  $D = 4$ ,  $N = 1$  string vacua,” Nucl. Phys. B **542** (1999) 112 [arXiv:hep-th/9808139].
- [21] G. Aldazabal, A. Font, L. E. Ibanez and G. Violero, “ $D = 4$ ,  $N = 1$ , type IIB orientifolds,” Nucl. Phys. B **536** (1998) 29 [arXiv:hep-th/9804026].
- [22] E. G. Gimon and J. Polchinski, “Consistency Conditions for Orientifolds and D-Manifolds,” Phys. Rev. D **54** (1996) 1667 [arXiv:hep-th/9601038].
- [23] C. Bachas, M. Bianchi, R. Blumenhagen, D. Lust and T. Weigand, “Comments on Orientifolds without Vector Structure,” JHEP **0808** (2008) 016 [arXiv:0805.3696 [hep-th]].  
E. Plauschinn, “The Generalized Green-Schwarz Mechanism for Type IIB Orientifolds with D3- and D7-Branes,” arXiv:0811.2804 [hep-th].  
T. W. Grimm and J. Louis, “The effective action of  $N = 1$  Calabi-Yau orientifolds,” Nucl. Phys. B **699** (2004) 387 [arXiv:hep-th/0403067].
- [24] P. S. Aspinwall, “Enhanced gauge symmetries and K3 surfaces,” Phys. Lett. B **357** (1995) 329 [arXiv:hep-th/9507012].
- [25] E. Palti, G. Tasinato and J. Ward, “WEAKLY-coupled IIA Flux Compactifications,” JHEP **0806** (2008) 084 [arXiv:0804.1248 [hep-th]].

- [26] M. Buican, D. Malyshev, D. R. Morrison, H. Verlinde and M. Wijnholt, “D-branes at singularities, compactification, and hypercharge,” *JHEP* **0701**, 107 (2007) [arXiv:hep-th/0610007].
- [27] J. P. Conlon, A. Maharana and F. Quevedo, “Towards Realistic String Vacua,” *JHEP* **0905**, 109 (2009) [arXiv:0810.5660 [hep-th]].
- [28] S. A. Abel, M. D. Goodsell, J. Jaeckel, V. V. Khoze and A. Ringwald, “Kinetic Mixing of the Photon with Hidden U(1)s in String Phenomenology,” *JHEP* **0807** (2008) 124 [arXiv:0803.1449 [hep-ph]].
- [29] E. G. Gimon and C. V. Johnson, “K3 Orientifolds,” *Nucl. Phys. B* **477**, 715 (1996) [arXiv:hep-th/9604129].